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Theoretical Evaluation of Atmospheric Drag Effects in the Motion of an Artificial Satellite

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The effect of atmospheric drag on the motion of an artificial satellite in the gravitational field of an oblate earth is evaluated. A spherical exponential model atmosphere is assumed together with the velocity-square law in drag acceleration. Atmospheric rotation is neglected. The method of canonical transformation is shown to be powerful even in the problem of nonconservative systems like the drag problem. The principal effect of the coupling between the nonsphericity of the earth and air drag is shown to be mixed secular terms and long-period terms produced by short-period terms of the drag-free problem. The result is complete up to the second order in case of secular and mixed secular terms and the first order in case of periodic terms if the drag coefficient A^* and the coefficient of the second harmonic of the earth potential k_2 are considered small quantities of the first order. The only exception is with long-period terms in the mean anomaly, for which the result is complete to the order A^*/k_2. The result is complete to the fourth power of the eccentricity in all cases, and is valid for all inclination except the critical inclination case.

INTRODUCTION

In most actual cases of artificial satellite motion, the effect of atmospheric drag is the most important next to the effect of oblateness. The usual approach of combining a theory of the oblateness perturbations, free from drag, with a theory of the drag effect on elliptic motion is clearly only a makeshift solution of the problem. The present paper is concerned with the problem of attending to oblateness perturbations and the drag effect in a single solution. The approach is essentially the familiar one of the method of the variation of arbitrary constants. The use of canonical variables has particular advantages in the treatment of this problem because of the ease with which the necessary transformation of variables can be made.

The development of the theory consists of five stages:

(i) Derivation of the equations of motion in the form expressed with Delaunay variables.

(ii) Derivation of the equations of motion in the form expressed with the canonical variables which are the solution of the drag-free problem.

(iii) Evaluation of drag acceleration in the new form of the equations of motion as functions of the doubly primed Delaunay variables.

(iv) Series development of the drag acceleration in powers of the eccentricity and in multiples of the mean anomaly.

(v) The integration of the equations of motion.

EQUATIONS OF MOTION FOR THE DELAUNAY VARIABLES

The equations of motion in an equatorial Cartesian system of coordinates are

$$\frac{d^2x_j}{dt^2}=\frac{\partial U}{\partial x_j}+X_j,\quad j=1,2,3,$$

or, in canonical form,

$$\frac{d\xi_j}{dt}=\frac{\partial F}{\partial \eta_j}+X_j,\quad \frac{d\eta_j}{dt}=-\frac{\partial F}{\partial \xi_j}\tag{1}$$

if

$$\xi_j=\dot{x}_j,\quad \eta_j=x_j,\quad \text{and}\quad F=-\tfrac{1}{2}\sum \dot{x}_j^2+U.$$

The force function U includes the oblateness terms and the terms X_j represent the drag acceleration.

It will be assumed that the drag acceleration on the instantaneous velocity is directly proportional to the cross-section area, the atmospheric density, and the square of the velocity V relative to the atmosphere, and inversely proportional to the mass. It will also be assumed that the atmospheric density may be represented by a spherical exponential model from the perigee height upward. The nonsphericity of the atmosphere should be considered in further refinements of the theory; this presents no essential difficulties.

Let ω be the angular velocity of the atmosphere, r the radius vector, and a the osculating semimajor axis of the orbit. The expression for X_j is then

$$X_j = -A^*C(\dot{x}_j+\delta_j)V\exp(-\alpha r)$$

with A^* and α constants; C is also a constant, being so chosen that $C\exp(-\alpha r)$ is of the order unity,

$$V^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]-2\omega(x_1\dot{x}_2-x_2\dot{x}_1)+\omega^2(x_1^2+x_2^2),$$

and

$$\delta_1=+\omega x_2,\quad \delta_2=-\omega x_1,\quad \delta_3=0.$$

In the following, the rotation velocity of the atmosphere will be neglected. This may be less significant than the omission of nonsphericity of the atmosphere. Therefore the expressions for X_j and V are reduced to

$$X_j=-AV\exp(-\alpha r)\xi_j,\quad A=A^*C,$$

$$V^2=\mu\left[\frac{2}{r}-\frac{1}{a}\right]. \tag{2}$$

Applying to Eqs. (1) the canonical transformation $(\xi_j,\eta_j)\rightarrow(L_j,l_j)$, we have

$$\frac{dL_j}{dt}=\frac{\partial F}{\partial l_j}+P_j,\quad \frac{dl_j}{dt}=-\frac{\partial F}{\partial L_j}-Q_j,\quad F=\frac{\mu^2}{2L_1^2}-\frac{\mu}{r}+U \tag{3}$$

with

$$P_j=\sum_k X_k\frac{\partial x_k}{\partial l_j},\quad Q_j=\sum_k X_k\frac{\partial x_k}{\partial L_j}, \tag{4}$$

where L_j, l_j $(j=1, 2, 3)$ are used for the Delaunay variables, L, G, H, l, g, h. The evaluation of P_j, Q_j leads to the results:

$$P_j=-AV\exp(-\alpha r)p_j,\quad p_j=\sum_k \xi_k\frac{\partial\eta_k}{\partial l_j},$$

$$Q_j=-AV\exp(-\alpha r)q_j,\quad q_j=\sum_k \xi_k\frac{\partial\eta_k}{\partial L_j}.$$

With the use of relations in Kepler motion, it is found that

$$\begin{aligned}
&p_1=L_1[(2a/r)-1], && q_1=2e\sin u+(2/e)(L_2/L_1)\sin f,\\
&p_2=L_2, && q_2=-(2/e)\sin f,\\
&p_3=L_3, && q_3=0,
\end{aligned} \tag{5}$$

in which the eccentric anomaly u, the true anomaly f, as well as a, e, r in the usual meaning are functions of the Delaunay variables. The relations between P_j, Q_j, and X_j will be proved later as a special case of a general canonical transformation.

EQUATIONS OF MOTION FOR THE CANONICAL VARIABLES WHICH ARE THE SOLUTION OF THE DRAG-FREE PROBLEM

Applying to Eqs. (3) the second canonical transformation $(L_j,l_j)\rightarrow(L_j'',l_j'')$, we have

$$\frac{dL_j''}{dt}=P_j'',\quad \frac{dl_j''}{dt}=-\frac{\partial F^{**}}{\partial L_j''}-Q_j'',\quad j=1, 2, 3, \tag{6}$$

$$F^{**}=F^{**}(L_1'',L_2'',L_3''),$$

with

$$\begin{aligned}
P_j''&=\sum_k P_k\frac{\partial l_k}{\partial l_j''}+\sum_k Q_k\frac{\partial L_k}{\partial l_j''},\\
Q_j''&=\sum_k P_k\frac{\partial l_k}{\partial L_j''}+\sum_k Q_k\frac{\partial L_k}{\partial L_j''}.
\end{aligned} \tag{7}$$

The canonical variables L_j'', l_j'' are the solution of the drag-free problem and

$$F^{**}(L_1'',L_2'',L_3''),\quad l_j(L_1'',L_2'',L_3'',l_1'',l_2'',l_3''),$$
$$L_j(L_1'',L_2'',L_3'',l_1'',l_2'',l_3'')$$

are supplied by the drag-free theory (Brouwer 1959), provided the inclination is not too near the critical inclination.

Let p_j'', q_j'' be related to P_j'', Q_j'' in the same manner as p_j, q_j are related to P_j, Q_j so that

$$p_j''=\sum_k p_k\frac{\partial l_k}{\partial l_j''}+\sum_k q_k\frac{\partial L_k}{\partial l_j''},$$

$$q_j''=\sum_k p_k\frac{\partial l_k}{\partial L_j''}+\sum_k q_k\frac{\partial L_k}{\partial L_j''}.$$

Then p_j'', q_j'' may be written

$$p_j''=p_j+\delta p_j,\quad q_j''=q_j+\delta q_j,$$

with

$$\begin{aligned}
\delta p_j&=\sum_k p_k\frac{\partial(l_k-l_k'')}{\partial l_j''}+\sum_k q_k\frac{\partial L_k}{\partial l_j''},\\
\delta q_j&=\sum_k p_k\frac{\partial l_k}{\partial L_j''}+\sum_k q_k\frac{\partial(L_k-L_k'')}{\partial L_j''},
\end{aligned} \tag{8}$$

δp_j, δq_j being of the order of the oblateness perturbations which are considered to be small quantities of the first order. Assuming the drag coefficient also a first-order small quantity at the most, p_j, q_j present in Eqs. (8) may be replaced by the expressions (5) with change of (L_j,l_j) to (L_j'',l_j'') if effects up to the second order are considered, as is the case.

A general proof of canonical transformations of the type Eqs. (4), (7) is easily given. Let

$$\frac{dx_j}{dt}=\frac{\partial F}{\partial y_j}+X_j,\quad \frac{dy_j}{dt}=-\frac{\partial F}{\partial x_j}-Y_j,\quad j=1,\cdots,n$$

be given, in which X_j, Y_j are arbitrary functions of the variables x_j, y_j, and t. Let

$$x_j=x_j(x_k',y_k'),\quad y_j=y_j(x_k',y_k')$$

be a canonical transformation and

$$\frac{dx_j'}{dt}=\frac{\partial F^*}{\partial y_j'}+X_j',\quad \frac{dy_j'}{dt}=-\frac{\partial F^*}{\partial x_j'}-Y_j'$$

with

$$F(x_j,y_j)=F^*(x_j',y_j')$$

be the transformed equations.

With use of the original equations we have

$$\sum_i\left(\frac{dx_j}{dt}\frac{\partial y_j}{\partial y_p'}-\frac{dy_j}{dt}\frac{\partial x_j}{\partial y_p'}\right)=\frac{\partial F^*}{\partial y_p'}+\sum_i\left(X_j\frac{\partial y_j}{\partial y_p'}+Y_j\frac{\partial x_j}{\partial y_p'}\right).$$

In the left-hand member, the substitution

$$\frac{dx_j}{dt}=\sum_k\left(\frac{\partial x_j}{\partial x_k'}\frac{dx_k'}{dt}+\frac{\partial x_j}{\partial y_k'}\frac{dy_k'}{dt}\right),$$

$$\frac{dy_j}{dt}=\sum_k\left(\frac{\partial y_j}{\partial x_k'}\frac{dx_k'}{dt}+\frac{\partial y_j}{\partial y_k'}\frac{dy_k'}{dt}\right),$$

yields

$$\sum_k\left([x_k',y_p']\frac{dx_k'}{dt}+[y_k',y_p']\frac{dy_k'}{dt}\right)=\frac{dx_p'}{dt}$$

since all the Lagrange brackets are zero except

$$[x_p',y_p']=+1.$$

We have then

$$\frac{dx_p'}{dt}=\frac{\partial F^*}{\partial y_p'}+\sum_i\left(X_j\frac{\partial y_j}{\partial y_p'}+Y_j\frac{\partial x_j}{\partial y_p'}\right),$$

or finally

$$X_p'=\sum_i\left(X_j\frac{\partial y_j}{\partial y_p'}+Y_j\frac{\partial x_j}{\partial y_p'}\right).$$

In the same way,

$$Y_p'=\sum_i\left(X_j\frac{\partial y_j}{\partial x_p'}+Y_j\frac{\partial x_j}{\partial x_p'}\right).$$

EVALUATION OF δp_j, δq_j

If $(p_j)''$, $(q_j)''$ stand for p_j, q_j defined by Eqs. (5) with the elements L_j, l_j replaced by L_j'', l_j'', Eqs. (8) may be written

$$\begin{aligned}\delta p_j&=\sum_k(p_k)''\frac{\partial(l_k-l_k'')}{\partial l_j''}+\sum_k(q_k)''\frac{\partial L_k}{\partial l_j''},\\ \delta q_j&=\sum_k(p_k)''\frac{\partial l_k}{\partial L_j''}+\sum_k(q_k)''\frac{\partial(L_k-L_k'')}{\partial L_j''}.\end{aligned}\tag{8'}$$

The evaluation of δp_j, δq_j with the use of the above equations and the solution for L_j, l_j in the drag-free theory is straightforward but laborious. By taking advantage of properties of canonical transformations the work can be greatly reduced.

The expressions for $(p_j)''$, $(q_j)''$ are, in the original form,

$$(p_j)''=\sum_k\xi_k''\frac{\partial\eta_k''}{\partial l_j''},\quad (q_j)''=\sum_k\xi_k''\frac{\partial\eta_k''}{\partial L_j''}.$$

Introducing the determining functions S_1, S_1^* which were defined in the drag-free theory, we have

$$\begin{aligned}l_j-l_j''&=-\frac{\partial}{\partial L_j''}(S_1+S_1^*),\\ L_j-L_j''&=+\frac{\partial}{\partial l_j''}(S_1+S_1^*),\end{aligned}\tag{9}$$

where it is permitted to replace l_j by l_j'' in S_1 and l_j' by l_j'' in S_1^*. Therefore, if partial derivatives with respect to l_j'' or L_j'' are expressed by subscripts,

$$\begin{aligned}\delta p_j&=\sum_k\sum_i\xi_i''\frac{\partial\eta_i''}{\partial l_k''}\left[-\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial L_k''}\right]+\sum_k\sum_i\xi_i''\frac{\partial\eta_i''}{\partial L_k''}\left[+\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial l_k''}\right]\\ &=-\sum_k\sum_i\xi_i''\frac{\partial\eta_i''}{\partial l_k''}\sum_p\left[\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\xi_p''}\frac{\partial\xi_p''}{\partial L_k''}+\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\eta_p''}\frac{\partial\eta_p''}{\partial L_k''}\right]\\ &\qquad+\sum_k\sum_i\xi_i''\frac{\partial\eta_i''}{\partial L_k''}\sum_p\left[\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\xi_p''}\frac{\partial\xi_p''}{\partial l_k''}+\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\eta_p''}\frac{\partial\eta_p''}{\partial l_k''}\right]\\ &=\sum_k\sum_i\sum_p\xi_i''\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\xi_p''}\left[-\frac{\partial\eta_i''}{\partial l_k''}\frac{\partial\xi_p''}{\partial L_k''}+\frac{\partial\eta_i''}{\partial L_k''}\frac{\partial\xi_p''}{\partial l_k''}\right]\\ &\qquad+\sum_k\sum_i\sum_p\xi_i''\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\eta_p''}\left[-\frac{\partial\eta_i''}{\partial l_k''}\frac{\partial\eta_p''}{\partial L_k''}+\frac{\partial\eta_i''}{\partial L_k''}\frac{\partial\eta_p''}{\partial l_k''}\right]\\ &=\sum_i\sum_p\xi_i''\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\xi_p''}\{\eta_i'',\xi_p''\}+\sum_i\sum_p\xi_i''\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\eta_p''}\{\eta_i'',\eta_p''\}\\ &=-\sum_i\xi_i''\frac{\partial(S_{1l_j''}+S_{1l_j''}^*)}{\partial\xi_i''}\end{aligned}$$

since all the Poisson brackets are zero except

$$\{\eta_i'',\xi_i''\}=-1.$$

In an analogous way, we have

$$\delta q_j=-\sum_i \xi_i''\frac{\partial(S_{1L_j''}+S_{1L_j''}{}^*)}{\partial \xi_i''}.$$

Introduction of the operator D defined by

$$D\equiv-\sum_j \xi_j''\frac{\partial}{\partial \xi_j''} \tag{10}$$

leads to simple expressions for δp_j and δq_j:

$$\delta p_j=D\left(\frac{\partial S_1}{\partial l_j''}+\frac{\partial S_1{}^*}{\partial l_j''}\right),\quad \delta q_j=D\left(\frac{\partial S_1}{\partial L_j''}+\frac{\partial S_1{}^*}{\partial L_j''}\right). \tag{11}$$

With use of a property of canonical transformations, it is easily verified that

$$DL_j''=-\sum_k \xi_k''\frac{\partial L_j''}{\partial \xi_k''}=-\sum_k \xi_k''\frac{\partial \eta_k''}{\partial l_j''}=-(p_j)'',$$

and (12)

$$Dl_j''=-\sum_k \xi_k''\frac{\partial l_j''}{\partial \xi_k''}=+\sum_k \xi_k''\frac{\partial \eta_k''}{\partial L_j''}=+(q_j)''.$$

An immediate consequence is

$$D(L_3''/L_2'')=0$$

or, if I stands for the orbital inclination,

$$DI''=0. \tag{13a}$$

By the definition of D, we also have

$$Dr''=0,$$

or,

$$Dr''=D\beta''=0, \tag{13b}$$

β being the latitude of the satellite. The meaning of double primes in Eqs. (13) is the same as that of $(p_j)''$ or $(q_j)''$.

We thus find

$$\delta p_1=3\frac{\mu^2k_2}{L^3}\biggl\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left[-\eta^{-3}+\frac{a^3}{r^3}\left(1-2\frac{a}{r}\right)\right]+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\frac{a^3}{r^3}\left(1-2\frac{a}{r}\right)\cos(2g+2f)\biggr\},$$

$$\delta p_2=\frac{\mu^2k_2}{G^3}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\frac{1}{3}\cos(2g+2f)+e\cos(2g+f)+\frac{1}{3}e\cos(2g+3f)\right]$$

$$+\frac{\mu^2k_2}{G^3}e(1-16\theta^2+15\theta^4)(1-5\theta^2)^{-1}\left[\frac{1}{8}e\cos 2g+\frac{1}{2}\cos(2g+f)\right],$$

$$\delta p_3=0,$$

$$\delta q_1=\frac{\mu^2k_2}{e^2L^3G}\biggl\{(-1+3\theta^2)\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+1\right)\sin 2f+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\biggl[\left(-\frac{a^2}{r^2}\eta^2-\frac{a}{r}+1\right)\sin 2g$$

$$+\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+4f)\biggr]\biggr\}+\frac{\mu^2k_2}{eL^3G}\frac{a}{r}\biggl\{(-1+3\theta^2)\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+4\right)\sin f$$

$$+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\biggl[\left(-\frac{a^2}{r^2}\eta^2-\frac{a}{r}+2\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+2\right)\sin(2g+3f)\biggr]\biggr\}$$

$$+\frac{\mu^2k_2}{eL^3G}\left[\frac{1}{4}(1-11\theta^2)-10\theta^4(1-5\theta^2)^{-1}\right]\left[\left(1-3\frac{a}{r}\right)e\sin 2g+\sin(2g+f)-\sin(2g-f)\right],$$

$$\delta q_2=-\frac{\mu^2k_2}{e^2L^2G^2}\biggl\{(-1+3\theta^2)\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+1\right)\sin 2f+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\biggl[\left(-\frac{a^2}{r^2}\eta^2-\frac{a}{r}+1\right)\sin 2g$$

$$+\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+4f)\biggr]\biggr\}-\frac{\mu^2k_2}{eL^2G^2}\frac{a}{r}\biggl\{(-1+3\theta^2)\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+4\right)\sin f \tag{14}$$

$$+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\biggl[\left(-\frac{a^2}{r^2}\eta^2-\frac{a}{r}+2\right)\sin(2g+f)+\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+2\right)\sin(2g+3f)\biggr]\biggr\}$$

$$-\frac{\mu^2k_2}{eL^2G^2}\left(1-\frac{a}{r}\right)\biggl\{(-1+3\theta^2)\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+1\right)\sin f+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\biggl[\left(-\frac{a^2}{r^2}\eta^2-\frac{a}{r}-1\right)\sin(2g+f)$$

$$+\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\Big]\Big]\Big\}-\frac{\mu^2k_2}{G^4}(-3+15\theta^2)\left[2(f-l)+e(\sin f-\sin u)+\frac{1}{e}(1-\eta)\sin f\right]$$

$$-\frac{\mu^2k_2}{G^4}(9-15\theta^2)\left[\frac{1}{3}\sin(2g+2f)+\frac{1}{2}e\sin(2g+f)+\frac{1}{6}e\sin(2g+3f)\right]-\frac{\mu^2k_2}{G^4}\left[-\frac{1}{2}+\frac{11}{2}\theta^2+20\theta^4(1-5\theta^2)^{-1}\right]\sin 2g$$

$$-\frac{\mu^2k_2}{G^4}e\left[\frac{1}{8}(1-33\theta^2)-25\theta^4(1-5\theta^2)^{-1}-50\theta^6(1-5\theta^2)^{-2}\right][\sin(2g+f)+\sin(2g-f)]$$

$$-\frac{\mu^2k_2}{eG^4}\left[\frac{1}{8}(1-33\theta^2)\eta^2-\frac{3}{8}\left(1-\frac{55}{3}\theta^2\right)+(-25\eta^2+35)\theta^4(1-5\theta^2)^{-1}-50(\eta^2-1)\theta^6(1-5\theta^2)^{-2}\right]$$

$$\times[\sin(2g+f)-\sin(2g-f)],$$

$$\delta q_3=6\frac{\mu^2k_2}{G^4}\theta\left[2(f-l)+e(\sin f-\sin u)+\frac{1}{e}(1-\eta)\sin f-\frac{1}{3}\sin(2g+2f)-\frac{1}{2}e\sin(2g+f)-\frac{1}{6}e\sin(2g+3f)\right]$$

$$-4\frac{\mu^2k_2}{G^4}\theta\left[\frac{11}{8}+10\theta^2(1-5\theta^2)^{-1}+25\theta^4(1-5\theta^2)^{-2}\right]e\sin(2g+f),$$

where double primes are omitted for simplicity of printing, and

$$\theta=H''/G'',\quad \eta=G''/L''.$$

The contributions of the fourth and third harmonics are obtained with use of $\Delta_4S_1{}^*$ and $\Delta_3S_1{}^*$ in the drag-free theory. The result is, double primes being omitted,

$$\Delta_4\delta p_2=-\frac{5}{12}\frac{\mu^2}{G^3}\frac{k_4}{k_2}e[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}][e\cos 2g+4\cos(2g+f)],$$

$$\Delta_3\delta p_2=\frac{1}{4}\frac{\mu}{G}\frac{A_{3,0}}{k_2}e\sin I\left[\sin g+\frac{2}{e}\sin(g+f)\right];$$

$$\Delta_4\delta q_1=-\frac{5}{6}\frac{\mu^2}{eL^3G}\frac{k_4}{k_2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\left[e\left(1-3\frac{a}{r}\right)\sin 2g+\sin(2g+f)-\sin(2g-f)\right],$$

$$\Delta_3\delta q_1=-\frac{1}{2}\frac{\mu}{L^3}\frac{A_{3,0}}{k_2}\frac{G}{e^2}\sin I\left[e\left(1-3\frac{a}{r}\right)\cos g-\cos(g-f)\right];$$

$$\Delta_4\delta q_2=-\frac{\mu^2}{G^4}\frac{k_4}{k_2}\left[\frac{5}{3}-5\theta^2-\frac{40}{3}\theta^4(1-5\theta^2)^{-1}\right]\sin 2g-\frac{\mu^2}{G^4}\frac{k_4}{k_2}e\left[-\frac{5}{12}(1-9\theta^2)+\frac{50}{3}\theta^4(1-5\theta^2)^{-1}+\frac{100}{3}\theta^6(1-5\theta^2)^{-2}\right]$$

$$\times[\sin(2g+f)+\sin(2g-f)]-\frac{\mu^2}{G^4}\frac{k_4}{k_2}\left[-\frac{5}{12}(1-9\theta^2)\eta^2+\frac{5}{4}(1-5\theta^2)+\left(\frac{50}{3}\eta^2-\frac{70}{3}\right)\theta^4(1-5\theta^2)^{-1}\right. \tag{14'}$$

$$\left.+\frac{100}{3}(\eta^2-1)\theta^6(1-5\theta^2)^{-2}\right][\sin(2g+f)-\sin(2g-f)],$$

$$\Delta_3\delta q_2=-\frac{1}{2}\frac{\mu}{G^2}\frac{A_{3,0}}{k_2}\left\{\frac{\sin I}{e}\left[2\cos g+\frac{1}{e}\cos(g-f)\right]+\frac{1}{\sin I}\theta^2\cos(g+f)\right\};$$

$$\Delta_4\delta q_3=5\frac{\mu^2}{G^4}\frac{k_4}{k_2}e\theta\left[1+\frac{16}{3}\theta^2(1-5\theta^2)^{-1}+\frac{40}{3}\theta^4(1-5\theta^2)^{-2}\right]\sin(2g+f),$$

$$\Delta_3\delta q_3=\frac{1}{2}\frac{\mu}{G^2}\frac{A_{3,0}}{k_2}\frac{\theta}{\sin I}\cos(g+f).$$

It should be noted that short-period terms, when operated on by D, produce secular and long-period terms. A typical example is the derivation of δp_1:

$$\frac{\partial S_1}{\partial l_1''}=\frac{\mu^2 k_2}{L''^3}\left[\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left(-\eta^{-3}+\frac{a''^3}{r''^3}\right)+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\frac{a''^3}{r''^3}\cos(2g''+2f'')\right]$$

has only short-period terms, while δp_1, which is derived by

$$\delta p_1=D(\partial S_1/\partial l_1''),$$

has both constant and long-period terms in addition to short-period terms, that is,

$$[p_1]_{\text{const}}=-6\frac{\mu^2 k_2}{L''^3}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left[\frac{a''^4}{r''^4}\right]_{\text{const}},$$

$$[p_1]_{\text{long-period}}=-6\frac{\mu^2 k_2}{L''^3}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\times\left[\frac{a''^4}{r''^4}\cos 2f\right]_{\text{const}}\cdot\cos 2g.$$

This fact is one important result of the coupling effect between drag and oblateness.

SERIES DEVELOPMENT IN POWERS OF THE ECCENTRICITY AND IN MULTIPLES OF THE MEAN ANOMALY

The equations of motion, Eqs. (6), may be written in the form

$$\begin{aligned} dL_j''/dt&=-AV\exp(-\alpha r)p_j\\ &\qquad -AV''\exp(-\alpha r'')\delta p_j,\\ dl_j''/dt&=-(\partial F^{**}/\partial L_j'')+AV\exp(-\alpha r)q_j\\ &\qquad +AV''\exp(-\alpha r'')\delta q_j. \end{aligned}\tag{15}$$

Owing to the presence of the factor $V\exp(-\alpha r)$ or $V''\exp(-\alpha r'')$ in the right-hand members of the equations, a series development in terms of the eccentricity and the mean anomaly is inevitable before integration. The terms

$$V\exp(-\alpha r)p_i,\quad V\exp(-\alpha r)q_j$$

may be developed in terms of e, l at first, and finally in terms of e'', l'' with use of the solutions of the drag-free problem, while the terms

$$V''\exp(-\alpha r'')\delta p_j,\quad V''\exp(-\alpha r'')\delta q_j$$

are ready to be expanded in terms of e'', l''.

If now

$$x=(r/a)-1,\quad x''=(r''/a'')-1\tag{16}$$

are introduced,

$$\begin{aligned} V&=(\mu/a)^{\frac{1}{2}}[2(a/r)-1]^{\frac{1}{2}}\\ &=(\mu/a)^{\frac{1}{2}}[(1-x)/(1+x)]^{\frac{1}{2}}\\ &=(\mu/a)^{\frac{1}{2}}(1-x+\tfrac{1}{2}x^2-\tfrac{1}{2}x^3+\tfrac{3}{8}x^4-\tfrac{3}{8}x^5), \end{aligned}$$

$$\begin{aligned} \exp(-\alpha r)&=\exp[-\alpha a(1+x)]\\ &=\exp(-\alpha a)\exp(-\alpha a x)\\ &=\exp(-\alpha a)\left(1-\alpha a x+\frac{\alpha^2a^2}{2}x^2-\frac{\alpha^3a^3}{6}x^3+\frac{a^4\alpha^4}{24}x^4-\frac{\alpha^5a^5}{120}x^5\right), \end{aligned}\tag{17}$$

$$\begin{aligned} V\exp(-\alpha r)=(\mu/a)^{\frac{1}{2}}\exp(-\alpha a)\Bigg[&1-(1+\alpha a)x+\left(\frac{1}{2}+\alpha a+\frac{1}{2}\alpha^2a^2\right)x^2-\left(\frac{1}{2}+\frac{1}{2}\alpha a+\frac{1}{2}\alpha^2a^2+\frac{1}{6}\alpha^3a^3\right)x^3\\ &+\left(\frac{3}{8}+\frac{1}{2}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{6}\alpha^3a^3+\frac{1}{24}\alpha^4a^4\right)x^4-\left(\frac{3}{8}+\frac{3}{8}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{24}\alpha^4a^4+\frac{1}{120}\alpha^5a^5\right)x^5\Bigg]. \end{aligned}\tag{18}$$

By the definition, x is a variable of the order e. Convergence of the above expansions depends on the magnitude of αae. Tentatively the expansion is carried to the fifth power of αax in order to make the solution complete to the fourth order. If αae is large enough to endanger the convergence, a remedy may be to use

$$\exp(-\alpha r)=\exp(-\alpha a)\left[A_0-A_1\alpha a x+\frac{A_2\alpha^2a^2}{2}x^2-\frac{A_3\alpha^3a^3}{6}x^3+\frac{A_4\alpha^4a^4}{24}x^4-\frac{A_5\alpha^5a^5}{120}x^5\right]\tag{17'}$$

in place of the Taylor expansion used above. The necessary change is:

$$\alpha^j\to A_j\alpha^j,\quad j=0,1,\cdots,5,$$

and may be done at the very end of the development of the theory since powers of α are preserved without any change throughout the theory. The numerical values of A_j may be best determined when a value of $\alpha a e$ is given in an actual case.

Although $V\exp(-\alpha r)p_j$, $V\exp(-\alpha r)q_j$ should be considered as functions of Delaunay variables, it is convenient to express them as functions of a, x, e, f, and I. The explicit expressions are

$$\begin{aligned}
V\exp(-\alpha r)p_1 &= V\exp(-\alpha r)L[2(a/r)-1] \\
&= \mu\exp(-\alpha a)\Big[1-(3+\alpha a)x+\Big(\frac{9}{2}+3\alpha a+\frac{1}{2}\alpha^2a^2\Big)x^2-\Big(\frac{11}{2}+\frac{9}{2}\alpha a+\frac{3}{2}\alpha^2a^2+\frac{1}{6}\alpha^3a^3\Big)x^3 \\
&\quad+\Big(\frac{51}{8}+\frac{11}{2}\alpha a+\frac{9}{4}\alpha^2a^2+\frac{1}{2}\alpha^3a^3+\frac{1}{24}\alpha^4a^4\Big)x^4-\Big(\frac{57}{8}+\frac{51}{8}\alpha a+\frac{11}{4}\alpha^2a^2+\frac{3}{4}\alpha^3a^3+\frac{1}{8}\alpha^4a^4+\frac{1}{120}\alpha^5a^5\Big)x^5\Big], \\
V\exp(-\alpha r)p_2 &= V\exp(-\alpha r)G \\
&= \mu\exp(-\alpha a)\eta\Big[1-(1+\alpha a)x+\Big(\frac{1}{2}+\alpha a+\frac{1}{2}\alpha^2a^2\Big)x^2-\Big(\frac{1}{2}+\frac{1}{2}\alpha a+\frac{1}{2}\alpha^2a^2+\frac{1}{6}\alpha^3a^3\Big)x^3 \\
&\quad+\Big(\frac{3}{8}+\frac{1}{2}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{6}\alpha^3a^3+\frac{1}{24}\alpha^4a^4\Big)x^4-\Big(\frac{3}{8}+\frac{3}{8}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{24}\alpha^4a^4+\frac{1}{120}\alpha^5a^5\Big)x^5\Big], \\
V\exp(-\alpha r)p_3 &= V\exp(-\alpha r)H \\
&= [V\exp(-\alpha r)p_2]\theta, \\
V\exp(-\alpha r)q_1 &= V\exp(-\alpha r)\Big(2e\sin u+\frac{2}{e}\eta\sin f\Big) \qquad (19) \\
&= \Big(\frac{\mu}{a}\Big)^{\frac{1}{2}}\exp(-\alpha a)\Big[1-(1+\alpha a)x+\Big(\frac{1}{2}+\alpha a+\frac{1}{2}\alpha^2a^2\Big)x^2-\Big(\frac{1}{2}+\frac{1}{2}\alpha a+\frac{1}{2}\alpha^2a^2+\frac{1}{6}\alpha^3a^3\Big)x^3 \\
&\quad+\Big(\frac{3}{8}+\frac{1}{2}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{6}\alpha^3a^3+\frac{1}{24}\alpha^4a^4\Big)x^4-\Big(\frac{3}{8}+\frac{3}{8}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{24}\alpha^4a^4+\frac{1}{120}\alpha^5a^5\Big)x^5\Big] \\
&\qquad\times\Big[\frac{2}{e}\Big(1+\frac{1}{2}e^2+\frac{3}{8}e^4\Big)\sin f+2e\Big(1+\frac{1}{2}e^2\Big)x\sin f\Big], \\
V\exp(-\alpha r)q_2 &= V\exp(-\alpha r)\Big(-\frac{2}{e}\sin f\Big) \\
&= -\Big(\frac{\mu}{a}\Big)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e}\sin f\Big[1-(1+\alpha a)x+\Big(\frac{1}{2}+\alpha a+\frac{1}{2}\alpha^2a^2\Big)x^2-\Big(\frac{1}{2}+\frac{1}{2}\alpha a+\frac{1}{2}\alpha^2a^2+\frac{1}{6}\alpha^3a^3\Big)x^3 \\
&\quad+\Big(\frac{3}{8}+\frac{1}{2}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{6}\alpha^3a^3+\frac{1}{24}\alpha^4a^4\Big)x^4-\Big(\frac{3}{8}+\frac{3}{8}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{24}\alpha^4a^4+\frac{1}{120}\alpha^5a^5\Big)x^5\Big], \\
V\exp(-\alpha r)q_3 &= 0.
\end{aligned}$$

If ψ stands for any of these functions, we have

$$\psi=\psi''+\Big(\frac{\partial\psi}{\partial a}\Big)''\delta a+\Big(\frac{\partial\psi}{\partial x}\Big)''\delta x+\Big(\frac{\partial\psi}{\partial e}\Big)''\delta e+\Big(\frac{\partial\psi}{\partial f}\Big)''\delta f+\Big(\frac{\partial\psi}{\partial I}\Big)''\delta I. \qquad (20)$$

The variations δa, δe, δI are directly available in the result of the drag-free theory, and the variations δx, δf should be expressed in terms of δe, δl.

$$x=(r/a)-1=-e\cos u,$$

$$\partial x/\partial e=-\cos f, \quad \partial x/\partial l=(e/\eta)\sin f,$$

therefore,

$$\begin{aligned}
\delta x&=(\partial x/\partial e)\delta e+(\partial x/\partial l)\delta l \\
&=-\cos f\delta e+(e/\eta)\sin f\delta l,
\end{aligned}$$

while,

$$\begin{aligned}
\delta f&=(\partial f/\partial e)\delta e+(\partial f/\partial l)\delta l \\
&=\eta^{-2}(2+e\cos f)\sin f\cdot\delta e+\eta^{-3}[1+(e^2/2) \\
&\qquad+2e\cos f+(e^2/2)\cos 2f]\delta l.
\end{aligned}$$

By the drag-free theory, on the other hand, we have

$$\delta a=\frac{2k_2}{a}\left[\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left(\frac{a^3}{r^3}-\eta^{-3}\right)+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\frac{a^3}{r^3}\cos(2g+2f)\right],$$

$$\delta e=\delta_{lg}e+\delta_{sh}e,$$

$$\delta_{lg}e=\frac{k_2}{a^2\eta^2}e\left\{\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right\}\cos 2g+\frac{1}{4}\frac{A_{3,0}}{k_2}\frac{\sin I}{a}\sin g,$$

$$\delta_{sh}e=\frac{k_2}{a^2e}\eta^2\left\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left(\frac{a^3}{r^3}-\eta^{-3}\right)+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left(\frac{a^3}{r^3}-\eta^{-4}\right)\cos(2g+2f)\right.$$
$$\left.-\frac{1}{3\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)[3e\cos(2g+f)+e\cos(2g+3f)]\right\},$$

$$\delta I=\delta_{lg}I+\delta_{sh}I,$$

$$\delta_{lg}I=-\frac{e}{\eta^2\tan I}\delta_{lg}e, \tag{21}$$

$$\delta_{sh}I=\frac{1}{2}\frac{k_2}{a^2\eta^4}\theta\sin I[3\cos(2g+2f)+3e\cos(2g+f)+e\cos(2g+3f)],$$

$$\delta l=\delta_s l+\delta_{lg}l+\delta_{sh}l,$$

$$\delta_s l=3\frac{k_2}{a^2\eta^3}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt,$$

$$\delta_{lg}l=\frac{k_2}{a^2\eta}\left\{\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right\}\sin 2g-\frac{1}{4}\frac{A_{3,0}}{k_2}\frac{\sin I}{ae}\eta\cos g,$$

$$\delta_{sh}l=-\frac{k_2}{a^2e\eta}\left\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+1\right)\sin f+\frac{1}{2}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\left(-\frac{a^2}{r^2}\eta^2-\frac{a}{r}+1\right)\sin(2g+f)\right.\right.$$
$$\left.\left.+\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+\frac{1}{3}\right)\sin(2g+3f)\right]\right\}.$$

With these formulas, an explicit expression for δx is

$$\delta x=-\cos f(\delta_{lg}e+\delta_{sh}e)+(e/\eta)\sin f(\delta_s l+\delta_{lg}l+\delta_{sh}l).$$

The following is the result:

$$\delta x=\delta_{\mathrm{I}}x+\delta_{\mathrm{II}}x+\delta_{\mathrm{III}}x,$$

$$\delta_{\mathrm{I}}x=\frac{e}{\eta}\sin f\,\delta_s l=3\frac{k_2}{a^2\eta^4}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)e\sin f\cdot nt,$$

$$\delta_{\mathrm{II}}x=-\cos f\,\delta_{lg}e+\frac{e}{\eta}\sin f\delta_{lg}l$$

$$=-\frac{k_2}{a^2\eta^2}\left\{\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right\}e\cos(2g+f)-\frac{1}{4}\frac{A_{3,0}}{k_2}\frac{\sin I}{a}\sin(g+f),$$

$$\delta_{\text{III}}x=-\cos f\,\delta_{sh}e+\frac{e}{\eta}\sin f\,\delta_{sh}l \tag{21'}$$

$$=-\frac{k_2}{a^2\eta^2}\Bigg(\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left[\frac{1}{e}\left(\frac{a^3}{r^3}\eta^4-\eta\right)\cos f+\frac{1}{2}\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+1\right)(1-\cos 2f)\right]$$

$$+\frac{1}{2}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left\{\frac{1}{e}\left(\frac{a^3}{r^3}\eta^4-1\right)[\cos(2g+f)+\cos(2g+3f)]-\frac{1}{2}\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+1\right)\cos 2g\right.$$

$$\left.+\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}-\frac{5}{3}\right)\cos(2g+2f)-\frac{1}{2}\left(\frac{a^2}{r^2}\eta^2+\frac{a}{r}+\frac{5}{3}\right)\cos(2g+4f)\right\}\Bigg).$$

Since f is to be considered as a function of e and l, we have in Eq. (20)

$$(\partial\psi/\partial f)''\delta f=[\partial\psi(f)/\partial l]''\delta l+[\partial\psi(f)/\partial e]''\delta e, \tag{22}$$

the latter part of which is absorbed in $(\partial\psi/\partial e)''\delta e$. It is understood that all the variables in the expressions for δa, δe, δl, δx are doubly primed.

The variations, δa, δe, δl, δx should be also developed in powers of x. The final result for these expressions is

$$\delta a=\frac{2k_2}{a}\left\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left(-\frac{3}{2}e^2-\frac{15}{8}e^4-3x+6x^2-10x^3+15x^4\right)+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)(1-3x+6x^2-10x^3+15x^4)\cos(2g+2f)\right\},$$

$$\delta_{sh}e=\frac{k_2}{a^2e\eta^2}\left\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left[-\frac{3}{2}e^2+\frac{9}{8}e^4-(3-6e^2)x+(6-12e^2)x^2-10x^3+15x^4\right]\right.$$

$$+\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Big[(-2e^2+e^4-(3-6e^2+3e^4)x+(6-12e^2)x^2-(10-20e^2)x^3+15x^4-21x^5)\cos(2g+2f)$$

$$\left.-e\cos(2g+f)-\frac{1}{3}e\cos(2g+3f)\Big]\right\},$$

$$\delta_{sh}l=-\frac{k_2}{a^2e\eta}\left\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)[3-e^2-(3-2e^2)x+(4-3e^2)x^2-5x^3+6x^4]\sin f\right.$$

$$+\frac{1}{2}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Big[(-1+e^2+(3-2e^2)x-(4-3e^2)x^2+5x^3-6x^4)\sin(2g+f)$$

$$\left.+\left(\frac{7}{3}-e^2-(3-2e^2)x+(4-3e^2)x^2-5x^3+6x^4\right)\sin(2g+3f)\Big]\right\},$$

$$\delta_{\text{III}}x=-\frac{k_2}{a^2\eta^2}\left\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\mathcal{A}+\frac{1}{2}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\mathcal{B}\right\},$$

$$\mathcal{A}=\left[\frac{3}{2}-\frac{1}{2}e^2-\left(\frac{3}{2}-e^2\right)x+\left(2-\frac{3}{2}e^2\right)x^2-\frac{5}{2}x^3+3x^4\right]+\left[-\frac{3}{2}+\frac{1}{2}e^2+\left(\frac{3}{2}-e^2\right)x-\left(2-\frac{3}{2}e^2\right)x^2+\frac{5}{2}x^3-3x^4\right]\cos 2f$$

$$+\left[-\frac{3}{2}e+\frac{9}{8}e^3-\frac{1}{e}(3-6e^2+3e^4)x+\frac{1}{e}(6-12e^2)x^2-\frac{1}{e}(10-20e^2)x^3+\frac{15}{e}x^4-\frac{21}{e}x^5\right]\cos f,$$

$$\mathcal{B}=\left[-2e+e^3-\frac{1}{e}(3-6e^2+3e^4)x+\frac{1}{e}(6-12e^2)x^2-\frac{1}{e}(10-20e^2)x^3+\frac{15}{e}x^4-\frac{21}{e}x^5\right]\cos(2g+f)$$

$$+\left[-2e+e^3-\frac{1}{e}(3-6e^2+3e^4)x+\frac{1}{e}(6-12e^2)x^2-\frac{1}{e}(10-20e^2)x^3+\frac{15}{e}x^4-\frac{21}{e}x^5\right]\cos(2g+3f)$$

$$+\left[-\frac{3}{2}+\frac{1}{2}e^2+\left(\frac{3}{2}-e^2\right)x-\left(2-\frac{3}{2}e^2\right)x^2+\frac{5}{2}x^3-3x^4\right]\cos 2g+\left[\frac{1}{3}-e^2-(3-2e^2)x+(4-3e^2)x^2\right.$$

$$\left.-5x^3+6x^4\right]\cos(2g+2f)+\left[-\frac{11}{6}+\frac{1}{2}e^2+\left(\frac{3}{2}-e^2\right)x-\left(2-\frac{3}{2}e^2\right)x^2+\frac{5}{2}x^3-3x^4\right]\cos(2g+4f)$$

It will be noticed that $\delta_{sh}l$ appears in the calculation of both l and g; when l and g are combined to $l+g$, th result is complete to the order e^4, though the above expression for $\delta_{sh}l$ has a divisor e.

With use of Eqs. (20), (22), we may write the result of expansion in the following way

$$V\exp(-\alpha r)\begin{Bmatrix}p_j\\q_j\end{Bmatrix}=\begin{Bmatrix}(p_j)+(p_j)_a+(p_j)_x+(p_j)_e+(p_j)_I\\(q_j)+(q_j)_a+(q_j)_x+(q_j)_e+(q_j)_I\end{Bmatrix},$$

$$V''\exp(-\alpha r'')\begin{Bmatrix}\delta p_j\\\delta q_j\end{Bmatrix}=\begin{Bmatrix}(\delta p_j)\\(\delta q_j)\end{Bmatrix},$$

where, for instance, (p_j) means $[V\exp(-\alpha r)p_j]''$ and $(p_j)_a$ means

$$\{(\partial/\partial a)[V\exp(-\alpha r)p_j]\delta a\}''.$$

The meaning of the other symbols will be clear without confusion. After a lengthy calculation, we find:

(I)

$$(p_1)=\mu\exp(-\alpha a)\left\{1+\left(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\right)e^2+\left(\frac{21}{64}+\frac{3}{8}\alpha a+\frac{9}{32}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{64}\alpha^4a^4\right)e^4\right.$$

$$+\left[(3+\alpha a)e+\left(\frac{3}{4}+\frac{3}{2}\alpha a+\frac{7}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\right)e^3\right]\cos l+\left[\left(\frac{15}{4}+2\alpha a+\frac{1}{4}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(-\frac{1}{16}+\frac{11}{12}\alpha a+\frac{7}{8}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{48}\alpha^4a^4\right)e^4\right]\cos 2l+\left(\frac{19}{4}+3\alpha a+\frac{5}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\right)e^3\cos 3l$$

$$\left.+\left(\frac{391}{64}+\frac{101}{24}\alpha a+\frac{35}{32}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{192}\alpha^4a^4\right)e^4\cos 4l\right\},$$

$$(p_1)_a=\frac{k_2}{a^2}\mu\exp(-\alpha a)\left\{\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)\left[(-6\alpha a-3\alpha^2a^2)e^2+\left(-\frac{99}{8}\alpha a-\frac{75}{8}\alpha^2a^2-\frac{21}{8}\alpha^3a^3-\frac{3}{8}\alpha^4a^4\right)e^4\right]\right.$$

$$\left.+\frac{1}{\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\left(-\frac{3}{4}\alpha a-\alpha^2a^2-\frac{1}{4}\alpha^3a^3\right)e^2+\left(\frac{3}{16}\alpha a+\frac{1}{4}\alpha^2a^2-\frac{1}{2}\alpha^3a^3-\frac{1}{4}\alpha^4a^4-\frac{1}{48}\alpha^5a^5\right)e^4\right]\cos 2g\right\},$$

$$(p_1)_x=\frac{k_2}{a^2}\mu\exp(-\alpha a)\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt(-1)\left\{\left[(9+3\alpha a)e+\left(\frac{63}{4}+9\alpha a+\frac{21}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\right)e^3\right]\sin l\right.$$

$$+\left[\left(\frac{45}{2}+12\alpha a+\frac{3}{2}\alpha^2a^2\right)e^2+\left(\frac{381}{16}+\frac{61}{4}\alpha a+\frac{33}{8}\alpha^2a^2+\frac{3}{4}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\right)e^4\right]\sin 2l$$

$$\left.+\left(\frac{171}{4}+27\alpha a+\frac{45}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\right)e^3\sin 3l+\left(\frac{1173}{16}+\frac{101}{2}\alpha a+\frac{105}{8}\alpha^2a^2+\frac{3}{2}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\right)e^4\sin 4l\right\}$$

$$+\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^4}\left\{\left(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\right.$$

$$\times\biggl[\biggl(\frac{3}{2}+2\alpha a+\frac{1}{2}\alpha^2a^2\biggr)e^2+\biggl(-\frac{3}{16}-\frac{1}{2}\alpha a+\frac{5}{8}\alpha^2a^2+\frac{1}{2}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\biggr)e^4\biggr]\cos2g+\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I$$

$$\times\biggl[\biggl(\frac{3}{2}+2\alpha a+\frac{1}{2}\alpha^2a^2\biggr)e+\biggl(-\frac{27}{16}-\frac{5}{2}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{2}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\biggr)e^3\biggr]\sin g\biggr\}$$

$$+\frac{k_2}{a^2}\mu\exp(-\alpha a)\biggl\{\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)\biggl[9+3\alpha a+\biggl(36+\frac{105}{4}\alpha a+\frac{15}{2}\alpha^2a^2+\frac{3}{4}\alpha^3a^3\biggr)e^2$$

$$+\biggl(\frac{5079}{64}+\frac{4239}{64}\alpha a+\frac{819}{32}a^2a^2+\frac{187}{32}a^3a^3+\frac{51}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\biggr)e^4\biggr]+\frac{1}{\eta^4}\biggl(\frac{3}{2}-\frac{3}{2}\theta^2\biggr)\biggl[\biggl(\frac{31}{8}+\frac{31}{8}\alpha a+\frac{13}{8}\alpha^2a^2+\frac{5}{24}\alpha^3a^3\biggr)e^2$$

$$+\biggl(\frac{1005}{256}+\frac{915}{256}\alpha a+\frac{283}{128}\alpha^2a^2+\frac{167}{128}\alpha^3a^3+\frac{223}{768}\alpha^4a^4+\frac{13}{768}\alpha^5a^5\biggr)e^4\biggr]\cos2g\biggr\},$$

$$(\delta p_1)=\frac{k_2}{a^2}\mu\exp(-\alpha a)\biggl\{\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)\biggl[-6+\biggl(-24-\frac{15}{2}\alpha a-\frac{3}{2}\alpha^2a^2\biggr)e^2+\biggl(-\frac{1863}{32}-\frac{495}{16}\alpha a-\frac{135}{16}\alpha^2a^2$$

$$-\frac{15}{16}\alpha^3a^3-\frac{3}{32}\alpha^4a^4\biggr)e^4\biggr]+\frac{1}{\eta^4}\biggl(\frac{3}{2}-\frac{3}{2}\theta^2\biggr)\biggl[\biggl(-\frac{27}{8}-\frac{9}{4}\alpha a-\frac{3}{8}\alpha^2a^2\biggr)e^2$$

$$+\biggl(-\frac{123}{32}-\frac{33}{8}\alpha a-\frac{9}{4}\alpha^2a^2-\frac{1}{2}\alpha^3a^3-\frac{1}{8}\alpha^4a^4\biggr)e^4\biggr]\cos2g\biggr\},$$

$$\frac{dL''}{dt}=-A\mu\exp(-\alpha a)\biggl\{1+\biggl(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\biggr)e^2+\biggl(\frac{21}{64}+\frac{3}{8}\alpha a+\frac{9}{32}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{64}\alpha^4a^4\biggr)e^4$$

$$+\biggl[(3+\alpha a)e+\biggl(\frac{3}{4}+\frac{3}{2}\alpha a+\frac{7}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\biggr)e^3\biggr]\cos l+\biggl[\biggl(\frac{15}{4}+2\alpha a+\frac{1}{4}\alpha^2a^2\biggr)e^2+\biggl(-\frac{1}{16}+\frac{11}{12}\alpha a+\frac{7}{8}\alpha^2a^2$$

$$+\frac{1}{4}\alpha^3a^3+\frac{1}{48}\alpha^4a^4\biggr)e^4\biggr]\cos2l+\biggl(\frac{19}{4}+3\alpha a+\frac{5}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\biggr)e^3\cos3l$$

$$+\biggl(\frac{391}{64}+\frac{101}{24}\alpha a+\frac{35}{32}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{192}\alpha^4a^4\biggr)e^4\cos4l\biggr\}$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)nt(-1)\biggl\{\biggl[(9+3\alpha a)e+\biggl(\frac{63}{4}+9\alpha a+\frac{21}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\biggr)e^3\biggr]\sin l$$

$$+\biggl[\biggl(\frac{45}{2}+12\alpha a+\frac{3}{2}\alpha^2a^2\biggr)e^2+\biggl(\frac{381}{16}+\frac{61}{4}\alpha a+\frac{33}{8}\alpha^2a^2+\frac{3}{4}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\biggr)e^4\biggr]\sin2l$$

$$+\biggl(\frac{171}{4}+27\alpha a+\frac{45}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\biggr)e^3\sin3l+\biggl(\frac{1173}{16}+\frac{101}{2}\alpha a+\frac{105}{8}\alpha^2a^2+\frac{3}{2}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\biggr)e^4\sin4l\biggr\}$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)\biggl[3+3\alpha a+\biggl(12+\frac{51}{4}\alpha a+3\alpha^2a^2+\frac{3}{4}\alpha^3a^3\biggr)e^2$$

$$+\biggl(\frac{1353}{64}+\frac{1467}{64}\alpha a+\frac{249}{32}\alpha^2a^2+\frac{73}{32}\alpha^3a^3+\frac{21}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\biggr)e^4\biggr]$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\bigg(\frac{3}{2}-\frac{3}{2}\theta^2\bigg)\frac{1}{\eta^4}\bigg[\bigg(\frac{1}{2}+\frac{7}{8}\alpha a+\frac{1}{4}\alpha^2a^2-\frac{1}{24}\alpha^3a^3\bigg)e^2$$

$$+\bigg(\frac{21}{256}-\frac{93}{256}\alpha a+\frac{27}{128}\alpha^2a^2+\frac{39}{128}\alpha^3a^3-\frac{65}{768}\alpha^4a^4-\frac{1}{256}\alpha^5a^5\bigg)e^4\bigg]\cos 2g$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^4}\bigg\{\bigg(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\bigg)\bigg[\bigg(\frac{3}{2}+2\alpha a+\frac{1}{2}\alpha^2a^2\bigg)e^2$$

$$+\bigg(-\frac{3}{16}-\frac{1}{2}\alpha a+\frac{5}{8}\alpha^2a^2+\frac{1}{2}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\bigg)e^4\bigg]\cos 2g+\frac{1}{4}\frac{aA_{3,0}}{k_2^2}\sin I\bigg[\bigg(\frac{3}{2}+2\alpha a+\frac{1}{2}\alpha^2a^2\bigg)e$$

$$+\bigg(-\frac{27}{16}-\frac{5}{2}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{2}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\bigg)e^3\bigg]\sin g\bigg\}.$$

(II)

$$(p_2)=\mu\exp(-\alpha a)\eta\bigg\{1+\bigg(-\frac{1}{4}+\frac{1}{4}\alpha^2a^2\bigg)e^2+\bigg(-\frac{3}{64}-\frac{3}{32}\alpha^2a^2+\frac{1}{64}\alpha^4a^4\bigg)e^4+\bigg[(1+\alpha a)e+\bigg(-\frac{1}{4}-\frac{1}{2}\alpha a+\frac{1}{8}\alpha^2a^2$$

$$+\frac{1}{8}\alpha^3a^3\bigg)e^3\bigg]\cos l+\bigg[\bigg(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\bigg)e^2+\bigg(-\frac{19}{48}-\frac{7}{12}\alpha a-\frac{1}{8}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{48}\alpha^4a^4\bigg)e^4\bigg]\cos 2l$$

$$+\bigg(\frac{3}{4}+\alpha a+\frac{3}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\bigg)e^3\cos 3l+\bigg(\frac{157}{192}+\frac{13}{12}\alpha a+\frac{15}{32}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{192}\alpha^4a^4\bigg)e^4\cos 4l\bigg\},$$

$$(p_2)_a=\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\bigg\{\bigg(-\frac{1}{2}+\frac{3}{2}\theta^2\bigg)\bigg[-3\alpha^2a^2e^2+\bigg(\frac{3}{8}\alpha a-\frac{27}{8}\alpha^2a^2-\frac{3}{8}\alpha^3a^3-\frac{3}{8}\alpha^4a^4\bigg)e^4\bigg]$$

$$+\frac{1}{\eta^4}\bigg(\frac{3}{2}-\frac{3}{2}\theta^2\bigg)\bigg[\bigg(\frac{1}{4}\alpha a-\frac{1}{4}\alpha^3a^3\bigg)e^2+\bigg(-\frac{5}{16}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{2}\alpha^3a^3-\frac{1}{12}\alpha^4a^4-\frac{1}{48}\alpha^5a^5\bigg)e^4\bigg]\cos 2g\bigg\},$$

$$(p_2)_x=\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\bigg(-\frac{1}{2}+\frac{3}{2}\theta^2\bigg)nt(-1)\bigg\{\bigg[(3+3\alpha a)e+\bigg(\frac{15}{4}+3\alpha a-\frac{3}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\bigg)e^3\bigg]\sin l$$

$$+\bigg[\bigg(\frac{9}{2}+6\alpha a+\frac{3}{2}\alpha^2a^2\bigg)e^2+\bigg(\frac{41}{16}+\frac{19}{4}\alpha a+\frac{9}{8}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\bigg)e^4\bigg]\sin 2l+\bigg(\frac{27}{4}+9\alpha a+\frac{27}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\bigg)e^3\sin 3l$$

$$+\bigg(\frac{157}{16}+13\alpha a+\frac{45}{8}\alpha^2a^2+\alpha^3a^3+\frac{1}{16}\alpha^4a^4\bigg)e^4\sin 4l\bigg\}+\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^3}\bigg\{\bigg(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]$$

$$-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\bigg)\bigg[\bigg(-\frac{1}{2}+\frac{1}{2}\alpha^2a^2\bigg)e^2+\bigg(\frac{5}{16}-\frac{7}{8}\alpha^2a^2+\frac{1}{16}\alpha^4a^4\bigg)e^4\bigg]\cos 2g$$

$$+\frac{1}{4}\frac{aA_{3,0}}{k_2^2}\sin I\bigg[\bigg(-\frac{1}{2}+\frac{1}{2}\alpha^2a^2\bigg)e+\bigg(\frac{13}{16}-\frac{11}{8}\alpha^2a^2+\frac{1}{16}\alpha^4a^4\bigg)e^3\bigg]\sin g\bigg\}$$

$$+\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\bigg\{\bigg(-\frac{1}{2}+\frac{3}{2}\theta^2\bigg)\bigg[3+3\alpha a+\bigg(\frac{9}{2}+\frac{21}{4}\alpha a+3\alpha^2a^2+\frac{3}{4}\alpha^3a^3\bigg)e^2+\bigg(\frac{393}{64}+\frac{447}{64}\alpha a+\frac{129}{32}\alpha^2a^2+\frac{43}{32}\alpha^3a^3$$

$$+\frac{21}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\Bigg)e^4\Bigg]+\frac{1}{\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Bigg[\Bigg(\frac{1}{8}-\frac{1}{8}\alpha a+\frac{3}{8}\alpha^2a^2+\frac{5}{24}\alpha^3a^3\Bigg)e^2$$

$$+\Bigg(-\frac{41}{256}+\frac{131}{256}\alpha a-\frac{87}{128}\alpha^2a^2-\frac{131}{384}\alpha^3a^3+\frac{93}{768}\alpha^4a^4+\frac{13}{768}\alpha^5a^5\Bigg)e^4\Bigg]\cos 2g\Bigg\},$$

$$(p_2)_e=\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^3}\Bigg\{\Bigg(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigg)\Bigg[-e^2+\Bigg(\frac{1}{4}-\frac{1}{4}\alpha^2a^2\Bigg)e^4\Bigg]\cos 2g$$

$$+\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\Bigg[-e+\Bigg(\frac{5}{4}-\frac{1}{4}\alpha^2a^2\Bigg)e^3\Bigg]\sin g\Bigg\}$$

$$+\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\Bigg\{\Bigg(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigg)\Bigg[\Bigg(\frac{3}{2}+\frac{3}{2}\alpha a\Bigg)e^2+\Bigg(\frac{3}{2}+\frac{27}{16}\alpha a+\frac{3}{4}\alpha^2a^2+\frac{3}{16}\alpha^3a^3\Bigg)e^4\Bigg]+\frac{1}{\eta^4}\Bigg(\frac{3}{2}-\frac{3}{2}\theta^2\Bigg)$$

$$\times\Bigg[\Bigg(\frac{1}{2}+\frac{1}{4}\alpha a\Bigg)e^2+\Bigg(-\frac{1}{12}-\frac{1}{24}\alpha a+\frac{1}{4}\alpha^2a^2+\frac{1}{18}\alpha^3a^3\Bigg)e^4\Bigg]\cos 2g\Bigg\},$$

$$(\delta p_2)=\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^3}\Bigg(\frac{3}{2}-\frac{3}{2}\theta^2\Bigg)\Bigg[\Bigg(-\frac{11}{24}+\frac{1}{3}\alpha a+\frac{1}{24}\alpha^2a^2\Bigg)e^2+\Bigg(\frac{1}{96}-\frac{1}{6}\alpha a-\frac{1}{6}\alpha^2a^2+\frac{1}{18}\alpha^3a^3+\frac{1}{288}\alpha^4a^4\Bigg)e^4\Bigg]\cos 2g$$

$$+\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^3}\Bigg\{\Bigg(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigg)\Bigg[(-1+2\alpha a)e^2$$

$$+\Bigg(-\frac{1}{2}-\frac{5}{4}\alpha a+\frac{1}{4}\alpha^3a^3\Bigg)e^4\Bigg]\cos 2g+\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\Bigg[\alpha ae+\Bigg(-\frac{3}{8}-\frac{13}{8}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\Bigg)e^3\Bigg]\sin g\Bigg\},$$

$$\frac{dG''}{dt}=-A\mu\exp(-\alpha a)\eta\Bigg\{1+\Bigg(-\frac{1}{4}+\frac{1}{4}\alpha^2a^2\Bigg)e^2+\Bigg(-\frac{3}{64}-\frac{3}{32}\alpha^2a^2+\frac{1}{64}\alpha^4a^4\Bigg)e^4+\Bigg[(1+\alpha a)e+\Bigg(-\frac{1}{4}-\frac{1}{2}\alpha a+\frac{1}{8}\alpha^2a^2$$

$$+\frac{1}{8}\alpha^3a^3\Bigg)e^3\Bigg]\cos l+\Bigg[\Bigg(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\Bigg)e^2+\Bigg(-\frac{19}{48}-\frac{7}{12}\alpha a-\frac{1}{8}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{48}\alpha^4a^4\Bigg)e^4\Bigg]\cos 2l$$

$$+\Bigg(\frac{3}{4}+\alpha a+\frac{3}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\Bigg)e^3\cos 3l+\Bigg(\frac{157}{192}+\frac{13}{12}\alpha a+\frac{15}{32}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{192}\alpha^4a^4\Bigg)e^4\cos 4l\Bigg\}$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\Bigg(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigg)nl(-1)\Bigg\{\Bigg[(3+3\alpha a)e+\Bigg(\frac{15}{4}+3\alpha a-\frac{3}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\Bigg)e^3\Bigg]\sin l$$

$$+\Bigg[\Bigg(\frac{9}{2}+6a\alpha+\frac{3}{2}\alpha^2a^2\Bigg)e^2+\Bigg(\frac{41}{16}+\frac{19}{4}\alpha a+\frac{9}{8}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\Bigg)e^4\Bigg]\sin 2l$$

$$+\Bigg(\frac{27}{4}+9\alpha a+\frac{27}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\Bigg)e^3\sin 3l+\Bigg(\frac{157}{16}+13\alpha a+\frac{45}{8}\alpha^2a^2+\alpha^3a^3+\frac{1}{16}\alpha^4a^4\Bigg)e^4\sin 4l\Bigg\}$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\Bigg(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigg)\Bigg[3+3\alpha a+\Bigg(6+\frac{27}{4}\alpha a+\frac{3}{4}\alpha^3a^3\Bigg)e^2$$

$$+\Bigg(\frac{489}{64}+\frac{579}{64}\alpha a+\frac{45}{32}\alpha^2a^2+\frac{37}{32}\alpha^3a^3-\frac{3}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\Bigg)e^4\Bigg]$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^3}\Bigl(\frac{3}{2}-\frac{3}{2}\theta^2\Bigr)\Bigl[\Bigl(\frac{1}{6}+\frac{17}{24}\alpha a+\frac{5}{12}\alpha^2a^2-\frac{1}{24}\alpha^3a^3\Bigr)e^2$$

$$+\Bigl(-\frac{179}{768}-\frac{7}{768}\alpha a-\frac{133}{384}\alpha^2a^2+\frac{311}{1152}\alpha^3a^3+\frac{95}{2304}\alpha^4a^4-\frac{1}{256}\alpha^5a^5\Bigr)e^4\Bigr]\cos 2g$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{1}{\eta^3}\Bigl\{\Bigl(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigr)$$

$$\times\Bigl[\Bigl(-\frac{5}{2}+2\alpha a+\frac{1}{2}\alpha^2a^2\Bigr)e^2+\Bigl(\frac{1}{16}-\frac{5}{4}\alpha a-\frac{9}{8}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\Bigr)e^4\Bigr]\cos 2g$$

$$+\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\Bigl[\Bigl(-\frac{3}{2}+\alpha a+\frac{1}{2}\alpha^2a^2\Bigr)e+\Bigl(\frac{27}{16}-\frac{13}{8}\alpha a-\frac{3}{2}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\Bigr)e^3\Bigr]\sin g\Bigr\}.$$

(III)

$$(p_3)=(p_2)\theta,\quad (p_3)_a=(p_2)_a\theta,\quad (p_3)_x=(p_2)_x\theta,\quad (p_3)_e=(p_2)_e\theta,$$

$$(p_3)_I=-\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{\theta}{\eta^3}\Bigl\{\Bigl(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigr)$$

$$\times\Bigl[-e^2+\Bigl(\frac{1}{4}-\frac{1}{4}\alpha^2a^2\Bigr)e^4\Bigr]\cos 2g+\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\Bigl[-e+\Bigl(\frac{5}{4}-\frac{1}{4}\alpha^2a^2\Bigr)e^3\Bigr]\sin g\Bigr\}$$

$$+\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{\theta}{\eta^3}\Bigl(\frac{3}{2}-\frac{3}{2}\theta^2\Bigr)\Bigl[\Bigl(\frac{3}{8}-\frac{1}{8}\alpha^2a^2\Bigr)e^2+\Bigl(-\frac{11}{96}+\frac{1}{12}\alpha a+\frac{1}{4}\alpha^2a^2-\frac{1}{36}\alpha^3a^3-\frac{1}{96}\alpha^4a^4\Bigr)e^4\Bigr]\cos 2g,$$

$$(\delta p_3)=0,$$

$$\frac{dH''}{dt}=-A\mu\exp(-\alpha a)\eta\theta\Bigl\{1+\Bigl(-\frac{1}{4}+\frac{1}{4}\alpha^2a^2\Bigr)e^2+\Bigl(-\frac{3}{64}-\frac{3}{32}\alpha^2a^2+\frac{1}{64}\alpha^4a^4\Bigr)e^4+\Bigl[(1+\alpha a)e+\Bigl(-\frac{1}{4}-\frac{1}{2}\alpha a+\frac{1}{8}\alpha^2a^2$$

$$+\frac{1}{8}\alpha^3a^3\Bigr)e^3\Bigr]\cos l+\Bigl[\Bigl(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\Bigr)e^2+\Bigl(-\frac{19}{48}-\frac{7}{12}\alpha a-\frac{1}{8}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{48}\alpha^4a^4\Bigr)e^4\Bigr]\cos 2l$$

$$+\Bigl(\frac{3}{4}+\alpha a+\frac{3}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\Bigr)e^3\cos 3l+\Bigl(\frac{157}{192}+\frac{13}{12}\alpha a+\frac{15}{32}\alpha^2a^2+\frac{1}{12}\alpha^3a^3+\frac{1}{192}\alpha^4a^4\Bigr)e^4\cos 4l\Bigr\}$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\theta\Bigl(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigr)nt(-1)\Bigl\{\Bigl[(3+3\alpha a)e+\Bigl(\frac{15}{4}+3\alpha a-\frac{3}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\Bigr)e^3\Bigr]\sin l$$

$$+\Bigl[\Bigl(\frac{9}{2}+6\alpha a+\frac{3}{2}\alpha^2a^2\Bigr)e^2+\Bigl(\frac{41}{16}+\frac{19}{4}\alpha a+\frac{9}{8}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\Bigr)e^4\Bigr]\sin 2l$$

$$+\Bigl(\frac{27}{4}+9\alpha a+\frac{27}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\Bigr)e^3\sin 3l+\Bigl(\frac{157}{16}+13\alpha a+\frac{45}{8}\alpha^2a^2+\alpha^3a^3+\frac{1}{16}\alpha^4a^4\Bigr)e^4\sin 4l\Bigr\}$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\eta\theta\Bigl(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigr)\Bigl[3+3\alpha a+\Bigl(6+\frac{27}{4}\alpha a+\frac{3}{4}\alpha^3a^3\Bigr)e^2$$

$$+\Bigl(\frac{489}{64}+\frac{579}{64}\alpha a+\frac{45}{32}\alpha^2a^2+\frac{37}{32}\alpha^3a^3-\frac{3}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\Bigr)e^4\Bigr]$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{\theta}{\eta^3}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\left(1+\frac{3}{8}\alpha a+\frac{1}{4}\alpha^2a^2-\frac{1}{24}\alpha^3a^3\right)e^2+\left(-\frac{275}{768}+\frac{185}{768}\alpha a+\frac{9}{128}\alpha^2a^2\right.\right.$$

$$\left.\left.+\frac{215}{1152}\alpha^3a^3+\frac{7}{256}\alpha^4a^4-\frac{1}{256}\alpha^5a^5\right)e^4\right]\cos2g$$

$$-A\frac{k_2}{a^2}\mu\exp(-\alpha a)\frac{\theta}{\eta^3}\left\{\left(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}(1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\left[\left(-\frac{1}{2}+\frac{1}{2}\alpha^2a^2\right)e^2\right.\right.$$

$$\left.\left.+\left(\frac{5}{16}-\frac{7}{8}\alpha^2a^2+\frac{1}{16}\alpha^4a^4\right)e^4\right]\cos2g+\frac{1}{4}\frac{aA_{3,0}}{k_2^2}\sin I\left[\left(-\frac{1}{2}+\frac{1}{2}\alpha^2a^2\right)e+\left(\frac{3}{16}-\frac{11}{8}\alpha^2a^2+\frac{1}{16}\alpha^4a^4\right)e^3\right]\sin g\right\}.$$

(IV)

$$(q_1)=\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta}\left\{\left[e+\left(-1+\frac{1}{8}\alpha^2a^2\right)e^3\right]\sin l+\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e^2+\left(-\frac{43}{24}-\frac{5}{8}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\right)e^4\right]\sin2l\right.$$

$$\left.+\left(2+\alpha a+\frac{1}{8}\alpha^2a^2\right)e^3\sin3l+\left(\frac{127}{48}+\frac{25}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{48}\alpha^3a^3\right)e^4\sin4l\right\}$$

$$-\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\left\{\frac{1}{2}\alpha ae^3\sin l+\left[e^2+\left(\frac{5}{8}+\frac{1}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^4\right]\sin2l+\left(2+\frac{1}{2}\alpha a\right)e^3\sin3l\right.$$

$$\left.+\left(\frac{25}{8}+\frac{5}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^4\sin4l\right\},$$

$$(q_1)_a=\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\left(\frac{3}{4}+\frac{5}{4}\alpha a+\frac{1}{2}\alpha^2a^2\right)e^2+\left(-\frac{1}{2}-\frac{21}{16}\alpha a-\frac{1}{8}\alpha^2a^2+\frac{19}{48}\alpha^3a^3+\frac{1}{24}\alpha^4a^4\right)e^4\right]\sin2g$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\left(\frac{1}{2}+\alpha a\right)e^2+\left(\frac{7}{8}+\frac{3}{4}\alpha a+\frac{13}{8}\alpha^2a^2+\frac{1}{4}\alpha^3a^3\right)e^4\right]\sin2g,$$

$$(q_1)_x=-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\left\{\left[\left(\frac{3}{2}+\frac{3}{2}\alpha a\right)e^2+\left(\frac{45}{16}+\frac{51}{16}\alpha a+\frac{15}{16}\alpha^2a^2+\frac{3}{16}\alpha^3a^3\right)e^4\right]\right.$$

$$+\left(\frac{15}{4}+\frac{9}{2}\alpha a+\frac{3}{4}\alpha^2a^2\right)e^3\cos l+\left[\left(-\frac{3}{2}-\frac{3}{2}\alpha a\right)e^2+\left(\frac{9}{2}+\frac{15}{2}\alpha a+\frac{3}{2}\alpha^2a^2\right)e^4\right]\cos2l$$

$$\left.+\left(-\frac{15}{4}-\frac{9}{2}\alpha a-\frac{3}{4}\alpha^2a^2\right)e^3\cos3l+\left(-\frac{117}{16}-\frac{147}{16}\alpha a-\frac{39}{16}\alpha^2a^2-\frac{3}{16}\alpha^3a^3\right)e^4\cos4l\right\}$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\left\{\left[3e^2+\left(\frac{63}{8}+\frac{15}{4}\alpha a+\frac{9}{8}\alpha^2a^2\right)e^4\right]+(9+3\alpha a)e^3\cos l\right.$$

$$\left.+\left[-3e^2+\left(\frac{21}{2}+6\alpha a\right)e^4\right]\cos2l+(-9-3\alpha a)e^3\cos3l+\left(\frac{147}{8}+\frac{39}{4}\alpha a+\frac{9}{8}\alpha^2a^2\right)e^4\cos4l\right\}$$

$$-\frac{k^2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left\{\left(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\sin2g\right.$$

$$\left.-\frac{1}{4}\frac{aA_{3,0}}{k_2^2}\frac{\eta^2\sin I}{e}\cos g\right\}\left[\left(\frac{1}{2}+\frac{1}{2}\alpha a\right)e^2+\left(-\frac{9}{16}-\frac{7}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^4\right]$$

$$+\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\Bigl\{\Bigl(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigr)\sin 2g$$

$$-\frac{1}{4}\frac{aA_{3,0}}{k_2^2}\frac{\eta^2\sin I}{e}\cos g\Bigr\}\Bigl[e^2+\Bigl(-\frac{3}{8}+\frac{5}{4}\alpha a+\frac{3}{8}\alpha^2a^2\Bigr)e$$

$$-\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\Bigl(\frac{3}{2}-\frac{3}{2}\theta^2\Bigr)\Bigl[\Bigl(\frac{7}{12}+\alpha a+\frac{5}{12}\alpha^2a^2\Bigr)e^2$$

$$+\Bigl(\frac{53}{192}-\frac{35}{384}\alpha a+\frac{137}{384}\alpha^2a^2+\frac{419}{1152}\alpha^3a^3+\frac{47}{1152}\alpha^4a^4\Bigr)e^4\Bigr]\sin 2g$$

$$+\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\Bigl(\frac{3}{2}-\frac{3}{2}\theta^2\Bigr)\Bigl[\Bigl(2+\frac{5}{3}\alpha a\Bigr)e^2+\Bigl(-\frac{35}{192}+\frac{137}{96}\alpha a+\frac{419}{192}\alpha^2a^2+\frac{47}{144}\alpha^3a^3\Bigr)e^4\Bigr]\sin 2g,$$

$$(q_1)_e=\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\Bigl(\frac{3}{2}-\frac{3}{2}\theta^2\Bigr)\Bigl[\frac{1}{4}+\Bigl(-\frac{17}{16}+\frac{1}{24}\alpha a+\frac{5}{48}\alpha^2a^2\Bigr)e^2$$

$$+\Bigl(-\frac{141}{512}-\frac{415}{384}\alpha a-\frac{293}{768}\alpha^2a^2+\frac{3}{128}\alpha^3a^3+\frac{3}{512}\alpha^4a^4\Bigr)e^4\Bigr]\sin 2g$$

$$-\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\Bigl(\frac{3}{2}-\frac{3}{2}\theta^2\Bigr)\Bigl[\frac{1}{2}+\Bigl(-\frac{25}{24}+\frac{1}{2}\alpha a+\frac{5}{24}\alpha^2a^2\Bigr)e^2$$

$$+\Bigl(-\frac{469}{256}-\frac{187}{96}\alpha a-\frac{39}{128}\alpha^2a^2+\frac{3}{32}\alpha^3a^3+\frac{3}{256}\alpha^4a^4\Bigr)e^4\Bigr]\sin 2g,$$

$$(q_1)_l=\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta}\Bigl(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigr)nt\Bigl\{\Bigl[\Bigl(\frac{3}{2}+\frac{3}{2}\alpha a\Bigr)e^2+\Bigl(\frac{45}{16}+\frac{51}{16}\alpha a+\frac{15}{16}\alpha^2a^2+\frac{3}{16}\alpha^3a^3\Bigr)e^4\Bigr]$$

$$+\Bigl[3e+\Bigl(\frac{21}{4}+\frac{9}{2}\alpha a+\frac{9}{8}\alpha^2a^2\Bigr)e^3\Bigr]\cos l+\Bigl[\Bigl(\frac{15}{2}+\frac{3}{2}\alpha a\Bigr)e^2+\Bigl(\frac{29}{4}+\frac{27}{4}\alpha a+\frac{9}{4}\alpha^2a^2+\frac{1}{4}\alpha^3a^3\Bigr)e^4\Bigr]\cos 2l$$

$$+\Bigl(\frac{57}{4}+\frac{9}{2}\alpha a+\frac{3}{8}\alpha^2a^2\Bigr)e^3\cos 3l+\Bigl(\frac{391}{16}+\frac{153}{16}\alpha a+\frac{21}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\Bigr)e^4\cos 4l\Bigr\}$$

$$-\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\Bigl(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigr)nt\Bigl\{\Bigl[3e^2+\Bigl(\frac{63}{8}+\frac{15}{4}\alpha a+\frac{9}{8}\alpha^2a^2\Bigr)e^4\Bigr]+\Bigl(9+\frac{9}{2}\alpha a\Bigr)e^3\cos l$$

$$+\Bigl[3e^2+\Bigl(15+9\alpha a+\frac{3}{2}\alpha^2a^2\Bigr)e^4\Bigr]\cos 2l+\Bigl(9+\frac{3}{2}\alpha a\Bigr)e^3\cos 3l+\Bigl(\frac{153}{8}+\frac{21}{4}\alpha a+\frac{3}{8}\alpha^2a^2\Bigr)e^4\cos 4l\Bigr\}$$

$$+\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\Bigl\{\Bigl(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigr)\sin 2g$$

$$-\frac{1}{4}\frac{aA_{3,0}}{k_2^2}\frac{\eta^2\sin I}{e}\cos g\Bigr\}\Bigl[\Bigl(\frac{1}{2}+\frac{1}{2}\alpha a\Bigr)e^2+\Bigl(-\frac{9}{16}-\frac{7}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\Bigr)e^4\Bigr]$$

$$-\frac{k_2}{a^2}\Bigl(\frac{\mu}{a}\Bigr)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\Bigl\{\Bigl(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigr)\sin 2g$$

$$-\frac{1}{4}\frac{aA_{3,0}}{k_2^2}\frac{\eta^2\sin I}{e}\cos g\Bigr\}\Bigl[e^2+\Bigl(-\frac{3}{8}+\frac{5}{4}\alpha a+\frac{3}{8}\alpha^2a^2\Bigr)e^4\Bigr]$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Bigg[\frac{1}{4}+\left(\frac{41}{48}+\frac{5}{8}\alpha a+\frac{1}{48}\alpha^2a^2\right)e^2$$

$$+\left(-\frac{517}{512}+\frac{185}{384}\alpha a+\frac{125}{256}\alpha^2a^2+\frac{79}{1152}\alpha^3a^3-\frac{5}{5608}\alpha^4a^4\right)e^4\Bigg]\sin 2g$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\left(\frac{15}{12}+\frac{1}{12}\alpha a\right)e^2+\left(\frac{305}{192}+\frac{383}{192}\alpha a+\frac{79}{192}\alpha^2a^2-\frac{5}{576}\alpha^3a^3\right)e^4\right]\sin 2g,$$

$$(\delta q_1)=-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Bigg[\frac{1}{2}+\left(\frac{1}{8}+\frac{3}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^2$$

$$+\left(-\frac{667}{768}-\frac{95}{192}\alpha a+\frac{241}{384}\alpha^2a^2+\frac{89}{576}\alpha^3a^3+\frac{11}{2304}\alpha^4a^4\right)e^4\Bigg]\sin 2g$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left\{\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right\}$$

$$\times\left[4+(-4+3\alpha a+\alpha^2a^2)e^2+\left(-\frac{9}{16}-\frac{27}{8}\alpha a-\frac{3}{4}\alpha^2a+\frac{3}{8}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left(-\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\right)\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e+\left(-\frac{81}{16}-\frac{29}{16}\alpha a+\frac{7}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^3\right]\cos g$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\frac{aA_{3,0}}{k_2{}^2}\sin I\left[\left(\frac{3}{4}+\frac{3}{4}\alpha a\right)e+\left(-\frac{51}{32}-\frac{51}{32}\alpha a+\frac{9}{32}\alpha^2a^2+\frac{3}{32}\alpha^3a^3\right)e^3\right]\cos g,$$

$$\frac{dl''}{dt}=-\frac{\partial F^{**}}{\partial L''}+A\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta}\Bigg\{\left[e+\left(-1+\frac{1}{8}\alpha^2a^2\right)e^3\right]\sin l+\Bigg[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e^2+\Bigg(-\frac{43}{24}-\frac{5}{8}\alpha a+\frac{1}{8}\alpha^2a^2$$

$$+\frac{1}{24}\alpha^3a^3\Bigg)e^4\Bigg]\sin 2l+\left(2+\alpha a+\frac{1}{8}\alpha^2a^2\right)e^3\sin 3l+\left(\frac{127}{48}+\frac{25}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{48}\alpha^3a^3\right)e^4\sin 4l\Bigg\}$$

$$-A\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\Bigg\{\frac{1}{2}\alpha ae^3\sin l+\left[e^2+\left(\frac{5}{8}+\frac{1}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^4\right]\sin 2l+\left(2+\frac{1}{2}\alpha a\right)e^3\sin 3l$$

$$+\left(\frac{25}{8}+\frac{5}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^4\sin 4l\Bigg\}$$

$$+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\Bigg\{\left[3e+\left(\frac{3}{2}+\frac{3}{8}\alpha^2a^2\right)e^3\right]\cos l$$

$$+\left[(9+3\alpha a)e^2+\left(\frac{11}{4}-\frac{3}{4}\alpha a+\frac{3}{4}\alpha^2a^2+\frac{1}{4}\alpha^3a^3\right)e^4\right]\cos 2l+\left(18+9\alpha a+\frac{9}{8}\alpha^2a^2\right)e^3\cos 3l$$

$$+\left(\frac{127}{4}+\frac{75}{4}\alpha a+\frac{15}{4}\alpha^2a^2+\frac{1}{4}\alpha^3a^3\right)e^4\cos 4l\Bigg\}$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)nt\Bigg\{\frac{3}{2}\alpha ae^3\cos l+\left[6e^2+\left(\frac{9}{2}+3\alpha a+\frac{3}{2}\alpha^2a^2\right)e^4\right]\cos 2l$$

$$+\left(18+\frac{9}{2}\alpha a\right)e^3\cos 3l+\left(\frac{3}{4}-\frac{9}{2}\alpha a-\frac{3}{4}\alpha^2a^2\right)e^4\cos 4l\Bigg\}$$

$$+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\frac{1}{2}+\left(-\frac{1}{24}+\frac{11}{12}\alpha a+\frac{5}{24}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(-\frac{1583}{768}-\frac{233}{128}\alpha a-\frac{3}{8}\alpha^2a^2+\frac{143}{1152}\alpha^3a^3+\frac{13}{2304}\alpha^4a^4\right)e^4\right]\sin 2$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\frac{1}{2}+\left(\frac{1}{8}+\frac{3}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(-\frac{667}{768}-\frac{95}{192}\alpha a+\frac{241}{384}\alpha^2a^2+\frac{49}{576}\alpha^3a^3+\frac{11}{2304}\alpha^4a^4\right)e^4\right]\sin 2$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\frac{1}{2}+\left(-\frac{31}{24}-\frac{1}{12}\alpha a+\frac{5}{24}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(\frac{625}{768}-\frac{121}{192}\alpha a-\frac{173}{384}\alpha^2a^2+\frac{5}{576}\alpha^3a^3+\frac{3}{256}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left\{\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right\}$$

$$\times\left[4+(-4+3\alpha a+\alpha^2a^2)e^2+\left(-\frac{9}{16}-\frac{27}{8}\alpha a-\frac{3}{4}\alpha^2a^2+\frac{3}{8}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^5}\left(-\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\right)\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e+\left(-\frac{81}{16}-\frac{29}{16}\alpha a+\frac{7}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^3\right]\cos g$$

$$+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\frac{aA_{3,0}}{k_2{}^2}\sin I\left[\left(\frac{3}{4}+\frac{3}{4}\alpha a\right)e+\left(-\frac{51}{32}-\frac{51}{32}\alpha a+\frac{9}{32}\alpha^2a^2+\frac{3}{32}\alpha^3a^3\right)e^3\right]\cos g.$$

(V)

$$(q_2)=-\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2}\left\{\left[e+\left(-1+\frac{1}{8}\alpha^2a^2\right)e^3\right]\sin l+\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e^2+\left(-\frac{43}{24}-\frac{5}{8}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\right)e^4\right]\sin 2l\right.$$

$$\left.+\left(2+\alpha a+\frac{1}{8}\alpha^2a^2\right)e^3\sin 3l+\left(\frac{127}{48}+\frac{25}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{48}\alpha^3a^3\right)e^4\sin 4l\right\},$$

$$(q_2)_a=-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\left(\frac{3}{4}+\frac{5}{4}\alpha a+\frac{1}{2}\alpha^2a^2\right)e^2+\left(-\frac{1}{2}-\frac{21}{16}\alpha a-\frac{1}{8}\alpha^2a^2+\frac{19}{48}\alpha^3a^3+\frac{1}{24}\alpha^4a^4\right)e^4\right]\sin 2g,$$

$$(q_2)_x=\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\left\{\left[\left(\frac{3}{2}+\frac{3}{2}\alpha a\right)e^2+\left(\frac{45}{16}+\frac{51}{16}\alpha a+\frac{15}{16}\alpha^2a^2+\frac{3}{16}\alpha^3a^3\right)e^4\right]\right.$$

$$+\left(\frac{15}{4}+\frac{9}{2}\alpha a+\frac{3}{4}\alpha^2a^2\right)e^3\cos l+\left[\left(-\frac{3}{2}-\frac{3}{2}\alpha a\right)e^2+\left(\frac{9}{2}+\frac{15}{2}\alpha a+\frac{3}{2}\alpha^2a^2\right)e^4\right]\cos 2l$$

$$\left.+\left(-\frac{15}{4}-\frac{9}{2}\alpha a-\frac{3}{4}\alpha^2a^2\right)e^3\cos 3l+\left(-\frac{117}{16}-\frac{147}{16}\alpha a-\frac{39}{16}\alpha^2a^2-\frac{3}{16}\alpha^3a^3\right)e^4\cos 4l\right\}$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^4}\Biggl\{\Biggl(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Biggr)\sin 2g$$

$$-\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\frac{\eta^2\sin I}{e}\cos g\Biggr\}\Biggl[\left(\frac{1}{2}+\frac{1}{2}\alpha a\right)e^2+\left(-\frac{9}{16}-\frac{7}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^4\Biggr]$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Biggl[\left(\frac{7}{12}+\alpha a+\frac{5}{12}\alpha^2a^2\right)e^2+\Biggl(\frac{53}{192}-\frac{35}{384}\alpha a+\frac{137}{384}\alpha^2a^2$$

$$+\frac{419}{1152}\alpha^3a^3+\frac{47}{1152}\alpha^4a^4\Biggr)e^4\Biggr]\sin 2g,$$

$$(q_2)_c=-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Biggl[\frac{1}{4}+\left(-\frac{17}{16}+\frac{1}{24}\alpha a+\frac{5}{48}\alpha^2a^2\right)e^2$$

$$+\left(-\frac{141}{512}-\frac{415}{384}\alpha a-\frac{293}{768}\alpha^2a^2+\frac{3}{128}\alpha^3a^3+\frac{3}{512}\alpha^4a^4\right)e^4\Biggr]\sin 2g,$$

$$(q_2)_l=-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\Biggl\{\Biggl[\left(\frac{3}{2}+\frac{3}{2}\alpha a\right)e^2+\left(\frac{45}{16}+\frac{51}{16}\alpha a+\frac{15}{16}\alpha^2a^2+\frac{3}{16}\alpha^3a^3\right)e^4\Biggr]$$

$$+\left[3e+\left(\frac{21}{4}+\frac{9}{2}\alpha a+\frac{9}{8}\alpha^2a^2\right)e^3\right]\cos l+\Biggl[\left(\frac{15}{2}+\frac{3}{2}\alpha a\right)e^2+\left(\frac{29}{4}+\frac{27}{4}\alpha a+\frac{9}{4}\alpha^2a^2+\frac{1}{4}\alpha^3a^3\right)e^4\Biggr]\cos 2l$$

$$+\left(\frac{57}{4}+\frac{9}{2}\alpha a+\frac{3}{8}\alpha^2a^2\right)e^3\cos 3l+\left(\frac{391}{16}+\frac{153}{16}\alpha a+\frac{21}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^4\cos 4l\Biggr\}$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^4}\Biggl\{\Biggl(\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Biggr)\sin 2g$$

$$-\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\frac{\eta^2\sin I}{e}\cos g\Biggr\}\Biggl[\left(\frac{1}{2}+\frac{1}{2}\alpha a\right)e^2+\left(-\frac{9}{16}-\frac{7}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^4\Biggr]$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Biggl[\frac{1}{4}+\left(\frac{41}{48}+\frac{15}{24}\alpha a+\frac{1}{48}\alpha^2a^2\right)e^2$$

$$+\left(-\frac{517}{512}+\frac{185}{384}a\alpha+\frac{375}{768}\alpha^2a^2+\frac{79}{1152}\alpha^3a^3-\frac{5}{4608}\alpha^4a^4\right)e^4\Biggr]\sin 2g,$$

$$(\delta q_2)=\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^6}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Biggl[\frac{1}{2}+\left(\frac{1}{8}+\frac{3}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^2$$

$$+\left(-\frac{667}{768}-\frac{95}{192}\alpha a+\frac{241}{384}\alpha^2a^2+\frac{49}{576}\alpha^3a^3+\frac{11}{2304}\alpha^4a^4\right)e^4\Biggr]\sin 2g$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\Biggl[\frac{1}{2}+\left(\frac{31}{24}+\frac{19}{12}\alpha a+\frac{1}{24}\alpha^2a^2\right)e^2$$

$$+\left(-\frac{187}{768}+\frac{49}{192}\alpha a+\frac{123}{128}\alpha^2a^2+\frac{97}{576}\alpha^3a^3-\frac{5}{2304}\alpha^4a^4\right)e^4\Biggr]\sin 2g$$

$$-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}(3-5\theta^2)\Biggl[\left(-\frac{5}{8}+\frac{1}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^2+\left(\frac{3}{32}-\frac{3}{16}\alpha a-\frac{5}{16}\alpha^2a^2+\frac{3}{16}\alpha^3a^3+\frac{1}{96}\alpha^4a^4\right)e^4\Biggr]\sin 2g$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left\{\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right\}$$

$$\times\left[4+(-1+\alpha^2a^2)e^2+\left(-\frac{3}{16}-\frac{3}{8}\alpha^2a^2+\frac{1}{16}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left\{\frac{1}{8}[1-33\theta^2-200\theta^4(1-5\theta^2)^{-1}-400\theta^6(1-5\theta^2)^{-2}]\right.$$

$$\left.-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-9\theta^2-40\theta^4(1-5\theta^2)^{-1}-80\theta^6(1-5\theta^2)^{-2}]\right\}\left[(1-\alpha a)e^2+\left(\frac{1}{8}+\frac{5}{8}\alpha a+\frac{1}{8}\alpha^2a^2-\frac{1}{8}\alpha^3a^3\right)e^4\right]\sin 2g$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^8}\left(-\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\right)\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e+\left(-\frac{81}{16}-\frac{29}{16}\alpha a+\frac{7}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^3\right]\cos g$$

$$+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\frac{aA_{3,0}}{k_2{}^2}\frac{\theta^2}{\sin I}\left[\left(\frac{1}{4}-\frac{1}{4}\alpha a\right)e+\left(-\frac{7}{32}+\frac{13}{32}\alpha a+\frac{1}{32}\alpha^2a^2-\frac{1}{32}\alpha^3a^3\right)e^3\right]\cos g,$$

$$\frac{dg''}{dt}=-\frac{\partial F^{**}}{\partial G''}-A\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2}\left\{\left[e+\left(-1+\frac{1}{8}\alpha^2a^2\right)e^3\right]\sin l+\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e^2+\left(-\frac{43}{24}-\frac{5}{8}\alpha a\right.\right.\right.$$

$$\left.\left.\left.+\frac{1}{8}\alpha^2a^2+\frac{1}{24}\alpha^3a^3\right)e^4\right]\sin 2l+\left(2+\alpha a+\frac{1}{8}\alpha^2a^2\right)e^3\sin 3l+\left(\frac{127}{48}+\frac{25}{16}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{48}\alpha^3a^3\right)e^4\sin 4l\right\}$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2}\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\left\{\left[3e+\left(\frac{3}{2}+\frac{3}{8}\alpha^2a^2\right)e^3\right]\cos l+\left[(9+3\alpha a)e^2+\left(\frac{11}{4}-\frac{3}{4}\alpha a\right.\right.\right.$$

$$\left.\left.\left.+\frac{3}{4}\alpha^2a^2+\frac{1}{4}\alpha^3a^3\right)e^4\right]\cos 2l+\left(18+9\alpha a+\frac{9}{8}\alpha^2a^2\right)e^3\cos 3l+\left(\frac{127}{4}+\frac{75}{4}\alpha a+\frac{15}{4}\alpha^2a^2+\frac{1}{4}\alpha^3a^3\right)e^4\cos 4l\right\}$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{e^2\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\frac{1}{2}+\left(-\frac{1}{24}+\frac{11}{12}\alpha a+\frac{5}{24}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(-\frac{1583}{768}-\frac{233}{128}\alpha a-\frac{3}{8}\alpha^2a^2+\frac{143}{1152}\alpha^3a^3+\frac{13}{2304}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^6}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\frac{1}{2}+\left(\frac{1}{8}+\frac{3}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(-\frac{667}{768}-\frac{95}{192}\alpha a+\frac{241}{384}\alpha^2a^2+\frac{49}{576}\alpha^3a^3+\frac{11}{2304}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left(\frac{3}{2}-\frac{3}{2}\theta^2\right)\left[\frac{1}{2}+\left(\frac{31}{24}+\frac{19}{12}\alpha a+\frac{1}{24}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(-\frac{187}{768}+\frac{49}{192}\alpha a+\frac{123}{128}\alpha^2a^2+\frac{97}{576}\alpha^3a^3-\frac{5}{2304}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}(3-5\theta^2)\left[\left(-\frac{5}{8}+\frac{1}{4}\alpha a+\frac{1}{8}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(\frac{3}{32}-\frac{3}{16}\alpha a-\frac{5}{16}\alpha^2a^2+\frac{3}{16}\alpha^3a^3+\frac{1}{96}\alpha^4a^4\right)e^4\right]\sin 2g$$

$$\begin{aligned}
&+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left\{\frac{1}{8}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right\}\\
&\qquad\qquad+\left[4+(-1+\alpha^2a^2)e^2+\left(-\frac{3}{16}-\frac{3}{8}\alpha^2a^2+\frac{1}{16}\alpha^4a^4\right)e^4\right]\sin 2g\\
&+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\left\{\frac{1}{8}[1-33\theta^2-200\theta^4(1-5\theta^2)^{-1}-400\theta^6(1-5\theta^2)^{-2}]\right.\\
&\qquad\left.-\frac{5}{12}\frac{k_4}{k_2{}^2}[1-9\theta^2-40\theta^4(1-5\theta^2)^{-1}-80\theta^6(1-5\theta^2)^{-2}]\right\}\left[(1-\alpha a)e^2+\left(\frac{1}{8}+\frac{5}{8}\alpha a+\frac{1}{8}\alpha^2a^2-\frac{1}{8}\alpha^3a^3\right)e^4\right]\sin 2g\\
&+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{2}{e^2\eta^8}\left(-\frac{1}{4}\frac{aA_{3,0}}{k_2{}^2}\sin I\right)\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e+\left(-\frac{81}{16}-\frac{29}{16}\alpha a+\frac{7}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^3\right]\cos g\\
&+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\frac{aA_{3,0}}{k_2{}^2}\frac{\theta^2}{\sin I}\left[\left(\frac{1}{4}-\frac{1}{4}\alpha a\right)e+\left(-\frac{7}{32}+\frac{13}{32}\alpha a+\frac{1}{32}\alpha^2a^2-\frac{1}{32}\alpha^3a^3\right)e^3\right]\cos g.
\end{aligned}$$

(VI)

$$\begin{aligned}
(\delta q_3)=&\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{\theta}{\eta^4}\left[\left(-\frac{5}{4}+\frac{1}{2}\alpha a+\frac{1}{4}\alpha^2a^2\right)e^2+\left(\frac{3}{16}-\frac{3}{8}\alpha a-\frac{5}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{48}\alpha^4a^4\right)e^4\right]\sin 2g\\
&+\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{\theta}{\eta^4}\left\{\frac{1}{8}[11+80\theta^2(1-5\theta^2)^{-1}+200\theta^4(1-5\theta^2)^{-2}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[12+64\theta^2(1-5\theta^2)^{-1}\right.\\
&\qquad\left.+160\theta^4(1-5\theta^2)^{-2}]\right\}\left[\left(\frac{1}{2}-\frac{1}{2}\alpha a\right)e^2+\left(\frac{1}{16}+\frac{5}{16}\alpha a+\frac{1}{16}\alpha^2a^2-\frac{1}{16}\alpha^3a^3\right)e^4\right]\sin 2g\\
&-\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\frac{aA_{3,0}}{k_2{}^2}\frac{\theta}{\sin I}\left[\left(\frac{1}{4}-\frac{1}{4}\alpha a\right)e+\left(-\frac{7}{32}+\frac{13}{32}\alpha a+\frac{1}{32}\alpha^2a^2-\frac{1}{32}\alpha^3a^3\right)e^3\right]\cos g,
\end{aligned}$$

$$\begin{aligned}
\frac{dh''}{dt}=&-\frac{\partial F^{**}}{\partial H''}+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{\theta}{\eta^4}\left[\left(-\frac{5}{4}+\frac{1}{2}\alpha a+\frac{1}{4}\alpha^2a^2\right)e^2+\left(\frac{3}{16}-\frac{3}{8}\alpha a-\frac{5}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{48}\alpha^4a^4\right)e^4\right]\sin 2g\\
&+A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{\theta}{\eta^4}\left\{\frac{1}{8}[11+80\theta^2(1-5\theta^2)^{-1}+200\theta^4(1-5\theta^2)^{-2}]-\frac{5}{12}\frac{k_4}{k_2{}^2}[12+64\theta^2(1-5\theta^2)^{-1}\right.\\
&\qquad\left.+160\theta^4(1-5\theta^2)^{-2}]\right\}\left[\left(\frac{1}{2}-\frac{1}{2}\alpha a\right)e^2+\left(\frac{1}{16}+\frac{5}{16}\alpha a+\frac{1}{16}\alpha^2a^2-\frac{1}{16}\alpha^3a^3\right)e^4\right]\sin 2g\\
&-A\frac{k_2}{a^2}\left(\frac{\mu}{a}\right)^{\frac{1}{2}}\exp(-\alpha a)\frac{1}{\eta^4}\frac{aA_{3,0}}{k_2{}^2}\frac{\theta}{\sin I}\left[\left(\frac{1}{4}-\frac{1}{4}\alpha a\right)e+\left(-\frac{7}{32}+\frac{13}{32}\alpha a+\frac{1}{32}\alpha^2a^2-\frac{1}{32}\alpha^3a^3\right)e^3\right]\cos g.
\end{aligned}$$

In the foregoing equations, all the variables in the right-hand members should be understood to be doubly primed.

The development is complete to e^4. The divisor e^2 in the right-hand members of the equations for dl''/dt and dg''/dt will be canceled in $d(l''+g'')/dt$; the divisor $\sin I$ in the right-hand members of the equations for dg''/dt and dh''/dt will be also canceled in $d(g''+h'')/dt$. As to the divisor, $1-5\theta^2$, the situation is different and the expansion may become invalid when the inclination is near the critical inclination, where the solution of the critical inclination (Hori 1960) may be used as a drag-free solution without any change of the method presented in this paper.

INTEGRATION

The method of successive approximation is applied. The zeroth approximation ($A=0$) corresponds to the solution of the drag-free problem:

$$L_j''=\text{const},\quad l_j''=-(\partial F^{**}/\partial L_j'')t+\text{const}.$$

In the first approximation, the above results are substituted in the right-hand members of the equations. At firs the equations for dL_j''/dt are integrated, then the equations for dl_j''/dt. In the terms $-\partial F^{**}/\partial L_j''$, since they c not contain the drag-coefficient A, the first-approximation results of L_j'' should be substituted except the par that contain k_2^2.

With use of the equations of dL_j''/dt with a, e, I constants and l, g linear functions of the time in the right-han members of them, the first approximation solution for L_j'' may be obtained after integration. By the drag-fre theory, we have

$$l=nt+\text{const}+O(k_2),\quad g=-\frac{3}{2}\cdot\frac{k_2}{a^2}\cdot\frac{1-5\theta^2}{\eta^4}nt+\text{const}+O(k_2^2).$$

Therefore,

$$\frac{L''}{L_0''}=1-Aa\exp(-\alpha a)\left\{\left[1+\left(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\right)e^2+\left(\frac{21}{64}+\frac{3}{8}\alpha a+\frac{9}{32}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{64}\alpha^4a^4\right)e^4\right]nt\right.$$

$$+\left[(3+\alpha a)e+\left(\frac{3}{4}+\frac{3}{2}\alpha a+\frac{7}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\right)e^3\right]\sin l+\left[\left(\frac{15}{8}+\alpha a+\frac{1}{8}\alpha^2a^2\right)e^2+\left(-\frac{1}{32}+\frac{11}{24}\alpha a+\frac{7}{16}\alpha^2a^2\right.\right.$$

$$\left.\left.+\frac{1}{8}\alpha^3a^3+\frac{1}{96}\alpha^4a^4\right)e^4\right]\sin 2l+\left(\frac{19}{12}+\alpha a+\frac{5}{24}\alpha^2a^2+\frac{1}{72}\alpha^3a^3\right)e^3\sin 3l$$

$$\left.+\left(\frac{391}{256}+\frac{101}{96}\alpha a+\frac{35}{128}\alpha^2a^2+\frac{1}{32}\alpha^3a^3+\frac{1}{768}\alpha^4a^4\right)e^4\sin 4l\right\}$$

$$-A\frac{k_2}{a^2}a\exp(-\alpha a)\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\left\{\left[(9+3\alpha a)e+\left(\frac{63}{4}+9\alpha a+\frac{21}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\right)e^3\right]\cos l+\left[\left(\frac{45}{4}+6\alpha a+\frac{3}{4}\alpha^2a^2\right)e^2\right.\right.$$

$$\left.+\left(\frac{381}{32}+\frac{61}{8}\alpha a+\frac{33}{16}\alpha^2a^2+\frac{3}{8}\alpha^3a^3+\frac{1}{32}\alpha^4a^4\right)e^4\right]\cos 2l+\left(\frac{171}{12}+9\alpha a+\frac{15}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\right)e^3\cos 3l$$

$$\left.+\left(\frac{1173}{64}+\frac{101}{8}\alpha a+\frac{105}{32}\alpha^2a^2+\frac{3}{8}\alpha^3a^3+\frac{1}{64}\alpha^4a^4\right)e^4\cos 4l\right\}$$

$$-A\frac{k_2}{a^2}a\exp(-\alpha a)\left(-\frac{1}{2}+\frac{3}{2}\theta^2\right)nt\left[3+3\alpha a+\left(12+\frac{51}{4}\alpha a+3a^2a^2+\frac{3}{4}\alpha^3a^3\right)e^2\right.$$

$$\left.+\left(\frac{1353}{64}+\frac{1467}{64}\alpha a+\frac{249}{32}\alpha^2a^2+\frac{73}{32}\alpha^3a^3+\frac{21}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\right)e^4\right]$$

$$+Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\left[\left(\frac{1}{4}+\frac{7}{18}\alpha a+\frac{1}{8}\alpha^2a^2-\frac{1}{48}\alpha^3a^3\right)e^2+\left(\frac{21}{512}-\frac{93}{512}\alpha a+\frac{27}{256}\alpha^2a^2+\frac{39}{256}\alpha^3a^3\right.\right.$$

$$\left.\left.-\frac{65}{1536}\alpha^4a^4-\frac{1}{512}\alpha^5a^5\right)e^4\right]\sin 2g$$

$$+Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\left\{\left(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\right.$$

$$\times\left[\left(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\right)e^2+\left(-\frac{3}{32}-\frac{1}{4}\alpha a+\frac{5}{16}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{32}\alpha^4a^4\right)e^4\right]\sin 2g-\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I$$

$$\left.\times\left[\left(\frac{3}{2}+2\alpha a+\frac{1}{2}\alpha^2a^2\right)e+\left(-\frac{27}{16}-\frac{5}{2}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{2}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\right)e^3\right]\cos g\right\},$$

$$
\begin{aligned}
\frac{G''}{G_0''} = {} & 1 - Aa\exp(-\alpha a)\biggl\{\biggl[1+\biggl(-\frac{1}{4}+\frac{1}{4}\alpha^2a^2\biggr)e^2+\biggl(-\frac{3}{64}-\frac{3}{32}\alpha^2a^2+\frac{1}{64}\alpha^4a^4\biggr)e^4\biggr]nt \\
& +\biggl[(1+\alpha a)e+\biggl(-\frac{1}{4}-\frac{1}{2}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\biggr)e^3\biggr]\sin l+\biggl[\biggl(\frac{3}{8}+\frac{1}{2}\alpha a+\frac{1}{8}\alpha^2a^2\biggr)e^2+\biggl(-\frac{19}{96}-\frac{7}{24}\alpha a-\frac{1}{16}\alpha^2a^2 \\
& +\frac{1}{24}\alpha^3a^3+\frac{1}{96}\alpha^4a^4\biggr)e^4\biggr]\sin 2l+\biggl(\frac{1}{4}+\frac{1}{3}\alpha a+\frac{1}{8}\alpha^2a^2+\frac{1}{72}\alpha^3a^3\biggr)e^3\sin 3l \\
& +\biggl(\frac{157}{768}+\frac{13}{48}\alpha a+\frac{15}{128}\alpha^2a^2+\frac{1}{48}\alpha^3a^3+\frac{1}{768}\alpha^4a^4\biggr)e^4\sin 4l\biggr\} \\
& -A\frac{k_2}{a^2}a\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)nt\biggl\{\biggl[(3+3\alpha a)e+\biggl(\frac{15}{4}+3\alpha a-\frac{3}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\biggr)e^3\biggr]\cos l \\
& +\biggl[\biggl(\frac{9}{4}+3\alpha a+\frac{3}{4}\alpha^2a^2\biggr)e^2+\biggl(\frac{41}{32}+\frac{19}{8}\alpha a+\frac{9}{16}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{32}\alpha^4a^4\biggr)e^4\biggr]\cos 2l \\
& +\biggl(\frac{9}{4}+3\alpha a+\frac{9}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\biggr)e^3\cos 3l+\biggl(\frac{157}{64}+\frac{13}{4}\alpha a+\frac{45}{32}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{64}\alpha^4a^4\biggr)e^4\cos 4l\biggr\} \\
& -A\frac{k_2}{a^2}a\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)nt\biggl[3+3\alpha a+\biggl(6+\frac{27}{4}\alpha a+\frac{3}{4}\alpha^3a^3\biggr)e^2 \\
& +\biggl(\frac{489}{64}+\frac{579}{64}\alpha a+\frac{45}{32}\alpha^2a^2+\frac{37}{32}\alpha^3a^3-\frac{3}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\biggr)e^4\biggr] \\
& +Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\biggl[\biggl(\frac{1}{12}+\frac{17}{48}\alpha a+\frac{5}{24}\alpha^2a^2-\frac{1}{48}\alpha^3a^3\biggr)e^2+\biggl(-\frac{179}{1536}-\frac{7}{1536}\alpha a-\frac{133}{768}\alpha^2a^2 \\
& +\frac{311}{2304}\alpha^3a^3+\frac{95}{4608}\alpha^4a^4-\frac{1}{512}\alpha^5a^5\biggr)e^4\biggr]\sin 2g \\
& +Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\biggl\{\biggl(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\biggr) \\
& \times\biggl[\biggl(-\frac{5}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\biggr)e^2+\biggl(\frac{1}{32}-\frac{5}{8}\alpha a-\frac{9}{16}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{32}\alpha^4a^4\biggr)e^4\biggr]\sin 2g-\frac{1}{6}\frac{aA_{3,0}}{k_2{}^2}\sin I \\
& \times\biggl[\biggl(-\frac{3}{2}+\alpha a+\frac{1}{2}\alpha^2a^2\biggr)e+\biggl(\frac{27}{16}-\frac{13}{8}\alpha a-\frac{3}{2}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{16}\alpha^4a^4\biggr)e^3\biggr]\cos g\biggr\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{H''}{H_0''} = {} & 1 - Aa\exp(-\alpha a)\biggl\{\biggl[1+\biggl(-\frac{1}{4}+\frac{1}{4}\alpha^2a^2\biggr)e^2+\biggl(-\frac{3}{64}-\frac{3}{32}\alpha^2a^2+\frac{1}{64}\alpha^4a^4\biggr)e^4\biggr]nt+\biggl[(1+\alpha a)e+\biggl(-\frac{1}{4}-\frac{1}{2}\alpha a+\frac{1}{8}\alpha^2a^2 \\
& +\frac{1}{8}\alpha^3a^3\biggr)e^3\biggr]\sin l+\biggl[\biggl(\frac{3}{8}+\frac{1}{2}\alpha a+\frac{1}{8}\alpha^2a^2\biggr)e^2+\biggl(-\frac{19}{96}-\frac{7}{24}\alpha a-\frac{1}{16}\alpha^2a^2+\frac{1}{24}\alpha^3a^3+\frac{1}{96}\alpha^4a^4\biggr)e^4\biggr]\sin 2l \\
& +\biggl(\frac{1}{4}+\frac{1}{3}\alpha a+\frac{1}{8}+\alpha^2a^2+\frac{1}{72}\alpha^3a^3\biggr)e^3\sin 3l+\biggl(\frac{157}{768}+\frac{13}{48}\alpha a+\frac{15}{128}\alpha^2a^2+\frac{1}{48}\alpha^3a^3+\frac{1}{768}\alpha^4a^4\biggr)e^4\sin 4l\biggr\}
\end{aligned}
$$

$$
\begin{aligned}
&-A\frac{k_2}{a^2}a\exp(-\alpha a)\Bigl(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigr)nt\Bigl\{\Bigl[(3+3\alpha a)e+\Bigl(\frac{15}{4}+3\alpha a-\frac{3}{8}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\Bigr)e^3\Bigr]\cos l+\Bigl[\Bigl(\frac{9}{4}+3\alpha a+\frac{3}{4}\alpha^2a^2\Bigr)e^2\\
&\quad+\Bigl(\frac{41}{32}+\frac{19}{8}\alpha a+\frac{9}{16}\alpha^2a^2+\frac{1}{8}\alpha^3a^3+\frac{1}{32}\alpha^4a^4\Bigr)e^4\Bigr]\cos 2l+\Bigl(\frac{9}{4}+3\alpha a+\frac{9}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\Bigr)e^3\cos 3l\\
&\quad+\Bigl(\frac{157}{64}+\frac{13}{4}\alpha a+\frac{45}{32}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{64}\alpha^4a^4\Bigr)e^4\cos 4l\Bigr\}-A\frac{k_2}{a^2}a\exp(-\alpha a)\Bigl(-\frac{1}{2}+\frac{3}{2}\theta^2\Bigr)nt\\
&\qquad\times\Bigl[3+3\alpha a+\Bigl(6+\frac{27}{4}\alpha a+\frac{3}{4}\alpha^3a^3\Bigr)e^2+\Bigl(\frac{489}{64}+\frac{579}{64}\alpha a+\frac{45}{32}\alpha^2a^2+\frac{37}{32}\alpha^3a^3-\frac{3}{64}\alpha^4a^4+\frac{3}{64}\alpha^5a^5\Bigr)e^4\Bigr]\\
&\quad+Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\Bigl[\Bigl(\frac{1}{2}+\frac{3}{16}\alpha a+\frac{1}{8}\alpha^2a^2-\frac{1}{48}\alpha^3a^3\Bigr)e^2+\Bigl(-\frac{275}{1536}+\frac{185}{1536}\alpha a+\frac{9}{256}\alpha^2a^2+\frac{215}{2304}\alpha^3a^3\\
&\qquad+\frac{7}{512}\alpha^4a^4-\frac{1}{512}\alpha^5a^5\Bigr)e^4\Bigr]\sin 2g\\
&\quad+Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\Bigl\{\Bigl(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Bigr)\\
&\qquad\times\Bigl[\Bigl(-\frac{1}{4}+\frac{1}{4}\alpha^2a^2\Bigr)e^2+\Bigl(\frac{5}{32}-\frac{7}{16}\alpha^2a^2+\frac{1}{32}\alpha^4a^4\Bigr)e^4\Bigr]\sin 2g-\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I\Bigl[\Bigl(-\frac{1}{2}+\frac{1}{2}\alpha^2a^2\Bigr)e\\
&\qquad+\Bigl(\frac{3}{16}-\frac{11}{8}\alpha^2a^2+\frac{1}{16}\alpha^4a^4\Bigr)e^3\Bigr]\cos g\Bigr\},
\end{aligned}
$$

where L_0'', G_0'', and H_0'' are three integration constants. Before proceeding to integration of l'', g'', and h'', the term $-\partial F^{**}/\partial L''$, $-\partial F^{**}/\partial G''$, $-\partial F^{**}/\partial H''$ should be evaluated with substitution of the above results in L'', G'', H'' included in them through a, e, I, n.

By the drag-free theory, we have

$$
\begin{aligned}
-\frac{\partial F^{**}}{\partial L''}=n\Bigl\{&1+\frac{3}{2}\frac{k_2}{a^2\eta^3}(-1+3\theta^2)+\frac{3}{32}\frac{k_2^2}{a^4\eta^7}[-15+16\eta+25\eta^2+(30-96\eta-90\eta^2)\theta^2+(105+144\eta+25\eta^2)\theta^4]\\
&+\frac{15}{16}\frac{k_4}{a^4\eta^7}e^2(3-30\theta^2-35\theta^4)\Bigr\},
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial F^{**}}{\partial G''}=n\Bigl\{&-\frac{3}{2}\frac{k_2}{a^2\eta^4}(1-5\theta^2)+\frac{3}{32}\frac{k_2^2}{a^4\eta^8}[-35+24\eta+25\eta^2+(90-192\eta-126\eta^2)\theta^2+(385+360\eta+45\eta^2)\theta^4]\\
&+\frac{5}{16}\frac{k_4}{a^4\eta^8}[21-91\eta^2+(-270+126\eta^2)\theta^2+(385-189\eta^2)\theta^4]\Bigr\},
\end{aligned}
$$

$$
-\frac{\partial F^{**}}{\partial H''}=n\Bigl\{-3\frac{k_2}{a^2\eta^4}\theta+\frac{3}{8}\frac{k_2^2}{a^4\eta^8}[(-5+12\eta+9\eta^2)\theta+(-35-36\eta-5\eta^2)\theta^3]+\frac{5}{4}\frac{k_4}{a^4\eta^8}\theta(3-7\theta^2)(5-3\eta^2)\Bigr\},
$$

where all the variables in the right-hand members should be understood as doubly primed, and functions of L'', G'', H''.

Let

$$
L''=L_0''(1+\Delta L),\quad G''=G_0''(1+\Delta G),\quad H''=H_0''(1+\Delta H),
$$

then,

$$n=\frac{\mu^2}{L''^3}=\frac{\mu^2}{L_0''^3}(1+\Delta L)^{-3}=n_0(1-3\Delta L)=n_0+n_0(-3\Delta L),$$

$$\begin{aligned}
n\frac{k_2}{a^2\eta^3}(-1+3\theta^2)&=\frac{\mu^2}{L''^3}\cdot\frac{\mu^2 k_2}{L''^4}\cdot\frac{L''^3}{G''^3}\left(-1+3\frac{H''^2}{G''^2}\right)\\
&=\frac{\mu^2}{L_0''^3}\cdot\frac{\mu^2 k_2}{L_0''^4}\cdot\frac{L_0''^3}{G_0''^3}(1-4\Delta L-3\Delta G)\left[-1+3\frac{H_0''^2}{G_0''^2}(1+2\Delta H-2\Delta G)\right]\\
&=n_0\frac{k_2}{a_0^2\eta_0^3}(-1+3\theta_0^2)(1-4\Delta L-3\Delta G)\left[1+\frac{6\theta_0^2}{-1+3\theta_0^2}(\Delta H-\Delta G)\right]\\
&=n_0\frac{k_2}{a_0^2\eta_0^3}(-1+3\theta_0^2)+n_0\frac{k_2}{a_0^2\eta_0^3}(-1+3\theta_0^2)(-4\Delta L-3\Delta G)+n_0\frac{k_2}{a_0^2\eta_0^3}6\theta_0^2(\Delta H-\Delta G),
\end{aligned}$$

$$n\frac{k_2}{a^2\eta^4}(1-5\theta^2)=n_0\frac{k_2}{a_0^2\eta_0^4}(1-5\theta_0^2)+n_0\frac{k_2}{a_0^2\eta_0^4}(1-5\theta_0^2)(-3\Delta L-4\Delta G)+n_0\frac{k_2}{a_0^2\eta_0^4}(-10\theta_0^2)(\Delta H-\Delta G),$$

$$n\frac{k_2}{a^2\eta^4}(-3\theta)=n_0\frac{k_2}{a_0^2\eta_0^4}(-3\theta_0)+n_0\frac{k_2}{a_0^2\eta_0^4}(-3\theta_0)[-3\Delta L-4\Delta G+(\Delta H-\Delta G)].$$

In the expression for n, the terms of the order A^*k_2 should be included as well as the terms of the order A^* in ΔL, while in the expressions for the others, only the terms of the order A^* are needed since the factor k_2 is multiplied.

With the results for ΔL, ΔG, ΔH, we obtain

$$\begin{aligned}
-4\Delta L-3\Delta G=Aa\exp(-\alpha a)\Bigg\{&\left[7+\left(\frac{9}{4}+4\alpha a+\frac{7}{4}\alpha^2a^2\right)e^2+\left(\frac{75}{64}+\frac{3}{2}\alpha a+\frac{27}{32}\alpha^2a^2+\frac{1}{2}\alpha^3a^3+\frac{7}{64}\alpha^4a^4\right)e^4\right]nt\\
&+\left[(15+7\alpha a)e+\left(\frac{9}{4}+\frac{9}{2}\alpha a+\frac{31}{8}\alpha^2a^2+\frac{7}{8}\alpha^3a^3\right)e^3\right]\sin l+\left[\left(\frac{69}{8}+\frac{11}{2}\alpha a+\frac{7}{8}\alpha^2a^2\right)e^2\right.\\
&\left.+\left(-\frac{23}{32}+\frac{23}{24}\alpha a+\frac{25}{16}\alpha^2a^2+\frac{5}{8}\alpha^3a^3+\frac{7}{96}\alpha^4a^4\right)e^4\right]\sin 2l+\left(\frac{85}{12}+5\alpha a+\frac{29}{24}\alpha^2a^2+\frac{7}{72}\alpha^3a^3\right)e^3\sin 3l\\
&+\left(\frac{1721}{256}+\frac{241}{48}\alpha a+\frac{185}{128}\alpha^2a^2+\frac{3}{16}\alpha^3a^3+\frac{7}{768}\alpha^4a^4\right)e^4\sin 4l\Bigg\}\\
&-Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\left[\left(\frac{5}{4}+\frac{377}{144}\alpha a+\frac{9}{8}\alpha^2a^2-\frac{7}{48}\alpha^3a^3\right)e^2+\left(-\frac{95}{512}-\frac{365}{512}\alpha a-\frac{25}{256}\alpha^2a^2\right.\right.\\
&\left.\left.+\frac{779}{768}\alpha^3a^3-\frac{165}{1536}\alpha^4a^4-\frac{7}{512}\alpha^5a^5\right)e^4\right]\sin 2g\\
&-Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\Bigg\{\left(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\\
&\times\left[\left(-\frac{3}{4}+7\alpha a+\frac{7}{4}\alpha^2a^2\right)e^2+\left(-\frac{9}{32}-\frac{23}{8}\alpha a-\frac{7}{16}\alpha^2a^2+\frac{11}{8}\alpha^3a^3+\frac{7}{32}\alpha^4a^4\right)e^4\right]\sin 2g-\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I\\
&\times\left[\left(\frac{3}{2}+11\alpha a+\frac{7}{2}\alpha^2a^2\right)e+\left(-\frac{27}{16}-\frac{119}{8}\alpha a-4\alpha^2a^2+\frac{19}{8}\alpha^3a^3+\frac{7}{16}\alpha^4a^4\right)e^3\right]\cos g\Bigg\},
\end{aligned}$$

$$
\begin{aligned}
-3\Delta L-4\Delta G=Aa\exp(-\alpha a)\Bigg\{&\Bigg[7+\left(\frac{5}{4}+3\alpha a+\frac{7}{4}\alpha^2a^2\right)e^2+\left(\frac{51}{64}+\frac{9}{8}\alpha a+\frac{15}{32}\alpha^2a^2+\frac{3}{8}\alpha^3a^3+\frac{7}{64}\alpha^4a^4\right)e^4\Bigg]nt\\
&+\left[(13+7\alpha a)e+\left(\frac{5}{4}+\frac{5}{2}\alpha a+\frac{25}{8}\alpha^2a^2+\frac{7}{8}\alpha^3a^3\right)e^3\right]\sin l+\Bigg[\left(\frac{57}{8}+5\alpha a+\frac{7}{8}\alpha^2a^2\right)e^2\\
&+\left(-\frac{85}{96}+\frac{5}{24}\alpha a+\frac{17}{16}\alpha^2a^2+\frac{13}{24}\alpha^3a^3+\frac{7}{96}\alpha^4a^4\right)e^4\Bigg]\sin 2l+\left(\frac{23}{4}+\frac{13}{3}\alpha a+\frac{9}{8}\alpha^2a^2+\frac{7}{72}\alpha^2a^2\right)e^3\sin 3l\\
&+\left(\frac{4147}{768}+\frac{407}{96}\alpha a+\frac{165}{128}\alpha^2a^2+\frac{17}{96}\alpha^3a^3+\frac{7}{768}\alpha^4a^4\right)e^4\sin 4l\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
-Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\Bigg[&\left(\frac{13}{12}+\frac{31}{12}\alpha a+\frac{29}{24}\alpha^2a^2-\frac{7}{48}\alpha^3a^3\right)e^2+\Bigg(-\frac{527}{1536}-\frac{865}{1536}\alpha a-\frac{451}{768}\alpha^2a^2\\
&+\frac{2297}{2304}\alpha^3a^3-\frac{205}{4608}\alpha^4a^4-\frac{7}{512}\alpha^5a^5\Bigg)e^4\Bigg]\sin 2g
\end{aligned}
$$

$$
\begin{aligned}
-Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\Bigg\{&\left(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\\
&\times\left[\left(-\frac{11}{4}+7\alpha a+\frac{7}{4}\alpha^2a^2\right)e^2+\left(-\frac{5}{32}-\frac{13}{4}\alpha a-\frac{21}{16}\alpha^2a^2+\frac{5}{4}\alpha^3a^3+\frac{7}{32}\alpha^4a^4\right)e^4\right]\sin 2g-\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I\\
&\times\left[\left(-\frac{3}{2}+10\alpha a+\frac{7}{2}\alpha^2a^2\right)e+\left(\frac{27}{16}-14\alpha a-\frac{45}{8}\alpha^2a^2+2\alpha^3a^3+\frac{7}{16}\alpha^4a^4\right)e^3\right]\cos g\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\Delta H-\Delta G=Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}&\left[\left(-\frac{1}{6}\alpha a-\frac{1}{12}\alpha^2a^2\right)e^2+\left(-\frac{1}{16}+\frac{1}{8}\alpha a+\frac{5}{24}\alpha^2a^2-\frac{1}{24}\alpha^3a^3-\frac{1}{144}\alpha^4a^4\right)e^4\right]\sin 2g\\
+Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\Bigg\{&\left(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\\
&\times\left[(1-\alpha a)e^2+\left(\frac{1}{8}+\frac{5}{8}\alpha a+\frac{1}{8}\alpha^2a^2-\frac{1}{8}\alpha^3a^3\right)e^4\right]\sin 2g-\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I\\
&\times\left[(1-\alpha a)e+\left(-\frac{3}{2}+\frac{13}{8}\alpha a+\frac{1}{8}\alpha^2a^2-\frac{1}{8}\alpha^3a^3\right)e^3\right]\cos g\Bigg\}.
\end{aligned}
$$

After integration we have as the first approximation solution for l'', g'', h'',

$$
\begin{aligned}
l''=l_0''+\Bigg\{&1+\frac{3}{2}\frac{k_2}{a^2\eta^3}(-1+3\theta^2)+\frac{3}{32}\frac{k_2^2}{a^4\eta^7}[-15+16\eta+25\eta^2+(30-96\eta-90\eta^2)\theta^2+(105+144\eta+25\eta^2)\theta^4]\\
&+\frac{15}{16}\frac{k_4}{a^4\eta^7}e^2(3-30\theta^2+35\theta^4)\Bigg\}nt+n\int(-3\Delta L)dt+\frac{3}{2}n\frac{k_2}{a^2\eta^3}(-1+3\theta^2)\int(-4\Delta L-3\Delta G)dt+9n\frac{k_2}{a^2\eta^3}\theta^2\int(\Delta H-\Delta G)dt\\
-Aa\exp(-\alpha a)\frac{2}{e^2\eta}\Bigg\{&\left[e+\left(-1+\frac{1}{8}\alpha^2a^2\right)e^3\right]\cos l+\left[\left(\frac{3}{4}+\frac{1}{4}\alpha a\right)e^2+\left(-\frac{43}{48}-\frac{5}{16}\alpha a+\frac{1}{16}\alpha^2a^2+\frac{1}{48}\alpha^3a^3\right)e^4\right]\cos 2l\\
&+\left(\frac{2}{3}+\frac{1}{3}\alpha a+\frac{1}{24}\alpha^2a^2\right)e^3\cos 3l+\left(\frac{127}{192}+\frac{25}{64}\alpha a+\frac{5}{64}\alpha^2a^2+\frac{1}{192}\alpha^3a^3\right)e^4\cos 4l\Bigg\}
\end{aligned}
$$

$$+Aa\exp(-\alpha a)\biggl\{\frac{1}{2}\alpha ae^3\cos l+\biggl[\frac{1}{2}e^2+\biggl(\frac{5}{16}+\frac{1}{8}\alpha a+\frac{1}{16}\alpha^2a^2\biggr)e^4\biggr]\cos 2l$$

$$+\biggl(\frac{2}{3}+\frac{1}{6}\alpha a\biggr)e^3\cos 3l+\biggl(\frac{25}{32}+\frac{5}{16}\alpha a+\frac{1}{32}\alpha^2a^2\biggr)e^4\cos 4l\biggr\}$$

$$+A\frac{k_2}{a^2}a\exp(-\alpha a)\frac{2}{e^2\eta}\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)nt\biggl\{\biggl[3e+\biggl(\frac{3}{2}+\frac{3}{8}\alpha^2a^2\biggr)e^3\biggr]\sin l+\biggl[\biggl(\frac{9}{2}+\frac{3}{2}\alpha a\biggr)e^2+\biggl(\frac{11}{8}-\frac{3}{8}\alpha a+\frac{3}{8}\alpha^2a^2$$

$$+\frac{1}{8}\alpha^3a^3\biggr)^{4}\biggr]\sin 2l+\biggl(6+3\alpha a+\frac{3}{8}\alpha^2a^2\biggr)e^3\sin 3l+\biggl(\frac{127}{16}+\frac{75}{16}\alpha a+\frac{15}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\biggr)e^4\sin 4l\biggr\}$$

$$-A\frac{k_2}{a^2}a\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)nt\biggl\{\frac{3}{2}\alpha ae^3\sin l+\biggl[3e^2+\biggl(\frac{9}{4}+\frac{3}{2}\alpha a+\frac{3}{4}\alpha^2a^2\biggr)e^4\biggr]\sin 2l$$

$$+\biggl(6+\frac{3}{2}\alpha a\biggr)e^3\sin 3l+\biggl(\frac{3}{16}-\frac{9}{8}\alpha a-\frac{3}{16}\alpha^2a^2\biggr)e^4\sin 4l\biggr\}$$

$$+Aa\exp(-\alpha a)\frac{2}{e^2\eta}\,\frac{1-\theta^2}{1-5\theta^2}\biggl[\frac{1}{4}+\biggl(-\frac{1}{48}+\frac{11}{24}\alpha a+\frac{5}{48}\alpha^2a^2\biggr)e^2+\biggl(-\frac{1583}{1536}-\frac{233}{256}\alpha a-\frac{3}{16}\alpha^2a^2$$

$$+\frac{143}{2304}\alpha^3a^3+\frac{13}{4608}\alpha^4a^4\biggr)e^4\biggr]\cos 2g$$

$$-Aa\exp(-\alpha a)\frac{2}{e^2\eta}\,\frac{1-\theta^2}{1-5\theta^2}\biggl[\frac{1}{4}+\biggl(\frac{1}{16}+\frac{3}{8}\alpha a+\frac{1}{16}\alpha^2a^2\biggr)e^2+\biggl(-\frac{667}{1536}-\frac{95}{384}\alpha a+\frac{241}{768}\alpha^2a^2$$

$$+\frac{49}{1152}\alpha^3a^3+\frac{11}{4608}\alpha^4a^4\biggr)e^4\biggr]\cos 2g$$

$$-Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\biggl[\frac{1}{4}+\biggl(-\frac{31}{48}-\frac{1}{24}\alpha a+\frac{5}{48}\alpha^2a^2\biggr)e^2+\biggl(\frac{625}{1536}-\frac{121}{384}\alpha a-\frac{173}{768}\alpha^2a^2+\frac{5}{1152}\alpha^3a^3+\frac{3}{512}\alpha^4a^4\biggr)e^4\biggr]\cos 2g$$

$$-Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\biggl\{\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\biggr\}$$

$$\times\biggl[2+\biggl(-2+\frac{3}{2}\alpha a+\frac{1}{2}\alpha^2a^2\biggr)e^2+\biggl(-\frac{9}{32}-\frac{27}{16}\alpha a-\frac{3}{8}\alpha^2a^2+\frac{3}{16}\alpha^3a^3+\frac{1}{32}\alpha^4a^4\biggr)e^4\biggr]\cos 2g$$

$$+Aa\exp(-\alpha a)\frac{2}{e^2\eta}(1-5\theta^2)^{-1}\biggl(-\frac{1}{6}\frac{aA_{3,0}}{k_2{}^2}\sin I\biggr)\biggl[\biggl(\frac{3}{2}+\frac{1}{2}\alpha a\biggr)e+\biggl(-\frac{81}{16}-\frac{29}{16}\alpha a+\frac{7}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\biggr)e^3\biggr]\sin g$$

$$-Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\frac{aA_{3,0}}{k_2{}^2}\sin I\biggl[\biggl(\frac{1}{2}+\frac{1}{2}\alpha a\biggr)e+\biggl(-\frac{17}{16}-\frac{17}{16}\alpha a+\frac{3}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\biggr)e^3\biggr]\sin g,$$

where

$$\begin{aligned}
n\int(-3\Delta L)dt={}&Aa\exp(-\alpha a)\Biggl\{\biggl[\frac{3}{2}+\biggl(\frac{9}{8}+\frac{3}{2}\alpha a+\frac{3}{8}\alpha^2a^2\biggr)e^2+\biggl(\frac{63}{128}+\frac{9}{16}\alpha a+\frac{27}{64}\alpha^2a^2+\frac{3}{16}\alpha^3a^3+\frac{3}{128}\alpha^4a^4\biggr)e^4\biggr](nt)^2\\
&+\biggl[(-9-3\alpha a)e+\biggl(-\frac{9}{4}-\frac{9}{2}\alpha a-\frac{21}{8}\alpha^2a^2-\frac{3}{8}\alpha^3a^3\biggr)e^3\biggr]\cos l+\biggl[\biggl(-\frac{45}{16}-\frac{3}{2}\alpha a-\frac{3}{16}\alpha^2a^2\biggr)e^2\\
&+\biggl(\frac{3}{64}-\frac{33}{48}\alpha a-\frac{21}{32}\alpha^2a^2-\frac{3}{16}\alpha^3a^3-\frac{1}{64}\alpha^4a^4\biggr)e^4\biggr]\cos 2l+\biggl(-\frac{19}{12}-\alpha a-\frac{5}{24}\alpha^2a^2-\frac{1}{72}\alpha^3a^3\biggr)e^3\cos 3l\\
&+\biggl(-\frac{1173}{1024}+\frac{101}{128}\alpha a+\frac{105}{512}\alpha^2a^2+\frac{3}{128}\alpha^3a^3+\frac{1}{1024}\alpha^4a^4\biggr)e^4\cos 4l\Biggr\}\\
&+A\frac{k_2}{a^2}a\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)nt\Biggl\{\biggl[(27+9\alpha a)e+\biggl(\frac{189}{4}+27\alpha a+\frac{63}{8}\alpha^2a^2+\frac{9}{8}\alpha^3a^3\biggr)e^3\biggr]\sin l\\
&+\biggl[\biggl(\frac{135}{8}+9\alpha a+\frac{9}{8}\alpha^2a^2\biggr)e^2+\biggl(\frac{1143}{64}+\frac{183}{16}\alpha a+\frac{99}{32}\alpha^2a^2+\frac{9}{16}\alpha^3a^3+\frac{3}{64}\alpha^4a^4\biggr)e^4\biggr]\sin 2l\\
&+\biggl(\frac{171}{12}+9\alpha a+\frac{15}{8}\alpha^2a^2+\frac{1}{8}\alpha^3a^3\biggr)e^3\sin 3l+\biggl(\frac{3519}{256}+\frac{303}{32}\alpha a+\frac{315}{128}\alpha^2a^2+\frac{9}{32}\alpha^3a^3+\frac{3}{256}\alpha^4a^4\biggr)e^4\sin 4l\Biggr\}\\
&+A\frac{k_2}{a^2}a\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)(nt)^2\biggl[\frac{3}{2}+\frac{3}{2}\alpha a+\biggl(6+\frac{51}{8}\alpha a+\frac{3}{2}\alpha^2a^2+\frac{3}{8}\alpha^3a^3\biggr)e^2\\
&+\biggl(\frac{1353}{128}+\frac{1467}{128}\alpha a+\frac{249}{64}\alpha^2a^2+\frac{73}{64}\alpha^3a^3+\frac{21}{128}\alpha^4a^4+\frac{3}{128}\alpha^5a^5\biggr)e^4\biggr]\\
&-A\frac{a^2}{k_2}a\exp(-\alpha a)\frac{1-\theta^2}{(1-5\theta^2)^2}\biggl[\biggl(\frac{1}{4}+\frac{7}{18}\alpha a+\frac{1}{8}\alpha^2a^2-\frac{1}{48}\alpha^3a^3\biggr)e^2+\biggl(-\frac{235}{512}-\frac{4421}{4608}\alpha a-\frac{37}{256}\alpha^2a^2\\
&+\frac{149}{768}\alpha^3a^3-\frac{65}{1536}\alpha^4a^4-\frac{1}{512}\alpha^5a^5\biggr)e^4\biggr]\cos 2g\\
&-A\frac{a^2}{k_2}a\exp(-\alpha a)(1-5\theta^2)^{-2}\Biggl\{\biggl(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\biggr)\\
&\times\biggl[\biggl(\frac{3}{4}+\alpha a+\frac{1}{4}\alpha^2a^2\biggr)e^2+\biggl(-\frac{51}{32}-\frac{9}{4}\alpha a-\frac{3}{16}\alpha^2a^2+\frac{1}{4}\alpha^3a^3+\frac{1}{32}\alpha^4a^4\biggr)e^4\biggr]\cos 2g+\frac{1}{6}\frac{aA_{3,0}}{k_2{}^2}\sin I\\
&\times\biggl[(3+4\alpha a+\alpha^2a^2)e+\biggl(-\frac{75}{8}-7\alpha a-\frac{7}{4}\alpha^2a^2+\alpha^3a^3+\frac{1}{8}\alpha^4a^4\biggr)e^3\biggr]\sin g\Biggr\},
\end{aligned}$$

$$\begin{aligned}
&\frac{3}{2}n\frac{k_2}{a^2\eta^3}(-1+3\theta^2)\int(-4\Delta L-3\Delta G)dt\\
&=A\frac{k_2}{a^2}a\exp(-\alpha a)\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)\Biggl\{\biggl[\frac{21}{2}+\biggl(\frac{153}{8}+6\alpha a+\frac{21}{8}\alpha^2a^2\biggr)e^2+\biggl(\frac{3393}{128}+\frac{45}{4}\alpha a+\frac{333}{64}\alpha^2a^2\\
&+\frac{3}{4}\alpha^3a^3+\frac{21}{128}\alpha^4a^4\biggr)e^4\biggr](nt)^2+\biggl[(-45-21\alpha a)e+\biggl(-\frac{297}{4}-45\alpha a-\frac{93}{8}\alpha^2a^2-\frac{21}{8}\alpha^3a^3\biggr)e^3\biggr]\cos l\\
&+\biggl[\biggl(-\frac{207}{16}-\frac{33}{4}\alpha a-\frac{21}{16}\alpha^2a^2\biggr)e^2+\biggl(-\frac{1095}{64}-\frac{221}{16}\alpha a-\frac{69}{16}\alpha^2a^2-\frac{15}{16}\alpha^3a^3-\frac{7}{64}\alpha^4a^4\biggr)e^4\biggr]\cos 2l
\end{aligned}$$

$$+\Big(-\frac{85}{12}-5\alpha a-\frac{29}{24}\alpha^2a^2-\frac{7}{72}\alpha^3a^3\Big)e^2\cos 3l+\Big(-\frac{5163}{1024}-\frac{241}{64}\alpha a-\frac{555}{512}\alpha^2a^2$$

$$-\frac{9}{64}\alpha^3a^3-\frac{7}{1024}\alpha^4a^4\Big)e^4\cos 4l\Big\}$$

$$-Aa\exp(-\alpha a)\Big(-\frac{1}{2}+\frac{3}{2}\theta^2\Big)\frac{1-\theta^2}{(1-5\theta^2)^2}\Big[\Big(\frac{5}{4}+\frac{377}{144}\alpha a+\frac{9}{8}\alpha^2a^2-\frac{7}{48}\alpha^3a^3\Big)e^2$$

$$+\Big(-\frac{415}{512}-\frac{9317}{4608}\alpha a-\frac{169}{256}\alpha^2a^2+\frac{241}{256}\alpha^3a^3-\frac{165}{1536}\alpha^4a^4-\frac{7}{512}\alpha^5a^5\Big)e^4\Big]\cos 2g$$

$$-Aa\exp(-\alpha a)\Big(-\frac{1}{2}+\frac{3}{2}\theta^2\Big)(1-5\theta^2)^{-2}\Big\{\Big(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]$$

$$-\frac{5}{18}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Big)\Big[\Big(-\frac{3}{4}+7\alpha a+\frac{7}{4}\alpha^2a^2\Big)e^2+\Big(\frac{3}{32}-\frac{51}{8}\alpha a-\frac{21}{16}\alpha^2a^2+\frac{11}{8}\alpha^3a^3$$

$$+\frac{7}{32}\alpha^4a^4\Big)e^4\Big]\cos 2g+\frac{1}{6}\frac{aA_{3,0}}{k_2{}^2}\sin I\Big[(3+22\alpha a+7\alpha^2a^2)e$$

$$+\Big(-\frac{39}{8}-\frac{163}{4}\alpha a-\frac{23}{2}\alpha^2a^2+\frac{19}{4}\alpha^3a^3+\frac{7}{8}\alpha^4a^4\Big)e^3\Big]\sin g\Big\},$$

$$9n\frac{k_2}{a^2\eta^3}\theta^2\int(\Delta H-\Delta G)dt$$

$$=Aa\exp(-\alpha a)\frac{\theta^2(1-\theta^2)}{(1-5\theta^2)^2}\Big[\Big(-\frac{1}{2}\alpha a-\frac{1}{4}\alpha^2a^2\Big)e^2+\Big(-\frac{3}{16}+\frac{5}{8}\alpha a+\frac{3}{4}\alpha^2a^2-\frac{1}{8}\alpha^3a^3-\frac{1}{48}\alpha^4a^4\Big)\Big]\cos 2g$$

$$+Aa\exp(-\alpha a)\frac{\theta^2}{(1-5\theta^2)^2}\Big\{\Big(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Big)$$

$$\times\Big[(3-3\alpha a)e^2+\Big(-\frac{9}{8}+\frac{27}{8}\alpha a+\frac{3}{8}\alpha^2a^2-\frac{3}{8}\alpha^3a^3\Big)e^4\Big]\cos 2g$$

$$+\frac{1}{6}\frac{aA_{3,0}}{k_2{}^2}\sin I\Big[(6-6\alpha a)e+\Big(-12+\frac{51}{4}\alpha a+\frac{3}{4}\alpha^2a^2-\frac{3}{4}\alpha^3a^3\Big)e^3\Big]\sin g\Big\},$$

l_0'' being a constant of integration.

As can be seen by the expression for $n\int(-3\Delta L)dt$, the accuracy of the long-period terms in l'' is limited to the order A^*/k_2. This is due to the presence of the term n without the drag coefficient A nor the coefficient of the oblateness k_2 in the right-hand member of the equation for dl''/dt through the term $-\partial F^{**}/\partial L''$. In order to obtain long-period terms to the order A^*, $k_2{}^2$ terms are needed for long-period terms of L in the drag-free theory, which are not available for the moment. In the drag-free problem, the situation may be the same if integration of long-period terms is performed with respect to t as is the case in the present paper, but the difficulty was avoided by performing the integration with respect to g with the help of the canonical transformation method (Brouwer 1959; Garfinkel 1959). This does not apply to the drag problem owing to the simple fact that the system is not conservative, although a method of canonical transformation can simplify the intermediate calculation needed.

In the cases of g'' and h'', long-period terms are complete up to the order A^* since the terms $-\partial F^{**}/\partial G''$ and $-\partial F^{**}/\partial H''$ have the oblateness coefficient k_2. The following is the result of the integration.

$$g''=g_0''+\Biggl\{-\frac{3}{2}\frac{k_2}{a^2\eta^4}(1-5\theta^2)+\frac{3}{32}\frac{k_2{}^2}{a^4\eta^8}[-35+24\eta+25\eta^2+(90-192\eta-126\eta^2)\theta^2+(385+360\eta+45\eta^2)\theta^4]$$

$$+\frac{5}{16}\frac{k_4}{a^4\eta^8}[21-91\eta^2+(-270+126\eta^2)\theta^2+(385-189\eta^2)\theta^4]\Biggr\}nt$$

$$-\frac{3}{2}n\frac{k_2}{a^2\eta^4}(1-5\theta^2)\int(-3\Delta L-4\Delta G)dt+15n\frac{k_2}{a^2\eta^4}\theta^2\int(\Delta H-\Delta G)dt$$

$$+Aa\exp(-\alpha a)\frac{2}{e^2}\Biggl\{\Biggl[e+\biggl(-1+\frac{1}{8}\alpha^2a^2\biggr)e^3\Biggr]\cos l+\Biggl[\biggl(\frac{3}{4}+\frac{1}{4}\alpha a\biggr)e^2+\biggl(-\frac{43}{48}-\frac{5}{16}\alpha a+\frac{1}{16}\alpha^2a^2+\frac{1}{48}\alpha^3a^3\biggr)e^4\Biggr]\cos 2l$$

$$+\biggl(\frac{2}{3}+\frac{1}{3}\alpha a+\frac{1}{24}\alpha^2a^2\biggr)e^3\cos 3l+\biggl(\frac{127}{192}+\frac{25}{64}\alpha a+\frac{5}{64}\alpha^2a^2+\frac{1}{192}\alpha^3a^3\biggr)e^4\cos 4l\Biggr\}$$

$$-A\frac{k_2}{a^2}a\exp(-\alpha a)\frac{2}{e^2}\biggl(-\frac{1}{2}+\frac{3}{2}\theta^2\biggr)nt\Biggl\{\Biggl[3e+\biggl(\frac{3}{2}+\frac{3}{8}\alpha^2a^2\biggr)e^3\Biggr]\sin l+\Biggl[\biggl(\frac{9}{2}+\frac{3}{2}\alpha a\biggr)e^2+\biggl(\frac{11}{8}-\frac{3}{8}\alpha a+\frac{3}{8}\alpha^2a^2$$

$$+\frac{1}{8}\alpha^3a^3\biggr)e^4\Biggr]\sin 2l+\biggl(6+3\alpha a+\frac{3}{8}\alpha^2a^2\biggr)e^3\sin 3l+\biggl(\frac{127}{16}+\frac{75}{16}\alpha a+\frac{15}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\biggr)e^4\sin 4l\Biggr\}$$

$$-Aa\exp(-\alpha a)\frac{2}{e^2}\frac{1-\theta^2}{1-5\theta^2}\Biggl[\frac{1}{4}+\biggl(-\frac{1}{48}+\frac{11}{24}\alpha a+\frac{5}{48}\alpha^2a^2\biggr)e^2$$

$$+\biggl(-\frac{1583}{1536}-\frac{233}{256}\alpha a-\frac{3}{16}\alpha^2a^2+\frac{143}{2304}\alpha^3a^3+\frac{13}{4608}\alpha^4a^4\biggr)e^4\Biggr]\cos 2g$$

$$+Aa\exp(-\alpha a)\frac{2}{e^2\eta^2}\frac{1-\theta^2}{1-5\theta^2}\Biggl[\frac{1}{4}+\biggl(\frac{1}{16}+\frac{3}{8}\alpha a+\frac{1}{16}\alpha^2a^2\biggr)e^2$$

$$+\biggl(-\frac{667}{1536}-\frac{95}{384}\alpha a+\frac{241}{768}\alpha^2a^2+\frac{49}{1152}\alpha^3a^3+\frac{11}{4608}\alpha^4a^4\biggr)e^4\Biggr]\cos 2g$$

$$-Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\Biggl[\frac{1}{4}+\biggl(\frac{31}{48}+\frac{19}{24}\alpha a+\frac{1}{48}\alpha^2a^2\biggr)e^2$$

$$+\biggl(-\frac{187}{1536}+\frac{49}{384}\alpha a+\frac{123}{256}\alpha^2a^2+\frac{97}{1152}\alpha^3a^3-\frac{5}{4608}\alpha^4a^4\biggr)e^4\Biggr]\cos 2g$$

$$-Aa\exp(-\alpha a)\biggl(1-\frac{5}{3}\theta^2\biggr)(1-5\theta^2)^{-1}\Biggl[\biggl(-\frac{5}{8}+\frac{1}{4}\alpha a+\frac{1}{8}\alpha^2a^2\biggr)e^2+\biggl(\frac{3}{32}-\frac{3}{16}\alpha a-\frac{5}{16}\alpha^2a^2+\frac{3}{16}\alpha^3a^3+\frac{1}{96}\alpha^4a^4\biggr)e^4\Biggr]\cos 2g$$

$$+Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\Biggl\{\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2{}^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\Biggr\}$$

$$\times\Biggl[2+\biggl(-\frac{1}{2}+\frac{1}{2}\alpha^2a^2\biggr)e^2+\biggl(-\frac{3}{32}-\frac{3}{16}\alpha^2a^2+\frac{1}{32}\alpha^4a^4\biggr)e^4\Biggr]\cos 2g$$

$$+Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\Biggl\{\frac{1}{12}[1-33\theta^2-200\theta^4(1-5\theta^2)^{-1}-400\theta^6(1-5\theta^2)^{-2}]$$

$$-\frac{5}{18}\frac{k_4}{k_2{}^2}[1-9\theta^2-40\theta^4(1-5\theta^2)^{-1}-80\theta^6(1-5\theta^2)^{-2}]\Biggr\}\Biggl[\biggl(\frac{1}{2}-\frac{1}{2}\alpha a\biggr)e^2+\biggl(\frac{1}{16}+\frac{5}{16}\alpha a+\frac{1}{16}\alpha^2a^2-\frac{1}{16}\alpha^3a^3\biggr)e^4\Biggr]\cos 2g$$

$$-Aa\exp(-\alpha a)\frac{2}{e^2\eta^4}(1-5\theta^2)^{-1}\left(-\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I\right)\left[\left(\frac{3}{2}+\frac{1}{2}\alpha a\right)e+\left(-\frac{81}{16}-\frac{29}{16}\alpha a+\frac{7}{16}\alpha^2a^2+\frac{1}{16}\alpha^3a^3\right)e^3\right]\sin g$$

$$-Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\frac{aA_{3,0}}{k_2^2}\frac{\theta^2}{\sin I}\left[\left(\frac{1}{6}-\frac{1}{6}\alpha a\right)e+\left(-\frac{7}{48}+\frac{13}{48}\alpha a+\frac{1}{48}\alpha^2a^2-\frac{1}{48}\alpha^3a^3\right)e^3\right]\sin g,$$

where

$$-\frac{3}{2}n\frac{k_2}{a^2\eta^4}(1-5\theta^2)\int(-3\Delta L-4\Delta G)dt$$

$$=-A\frac{k_2}{a^2}a\exp(-\alpha a)(1-5\theta^2)\left\{\left[\frac{21}{4}+\left(\frac{183}{16}+\frac{9}{4}\alpha a+\frac{21}{16}\alpha^2a^2\right)e^2+\left(\frac{4665}{256}+\frac{171}{32}\alpha a+\frac{381}{128}\alpha^2a^2+\frac{9}{32}\alpha^3a^3\right.\right.\right.$$

$$\left.\left.+\frac{21}{256}\alpha^4a^4\right)e^4\right](nt)^2+\left[\left(-\frac{39}{2}-\frac{21}{2}\alpha a\right)e+\left(-\frac{327}{8}-\frac{99}{4}\alpha a-\frac{75}{16}\alpha^2a^2-\frac{21}{16}\alpha^3a^3\right)e^3\right]\cos l+\left[\left(-\frac{171}{32}-\frac{15}{4}\alpha a\right.\right.$$

$$\left.\left.-\frac{21}{32}\alpha^2a^2\right)e^2+\left(-\frac{3849}{384}-\frac{245}{32}\alpha a-\frac{135}{64}\alpha^2a^2-\frac{13}{32}\alpha^3a^3-\frac{7}{128}\alpha^4a^4\right)e^4\right]\cos 2l+\left(-\frac{23}{8}-\frac{13}{6}\alpha a-\frac{9}{16}\alpha^2a^2-\frac{7}{144}\alpha^3a^3\right)e^3$$

$$\left.\times\cos 3l+\left(-\frac{4147}{2048}-\frac{407}{256}\alpha a-\frac{495}{1024}\alpha^2a^2-\frac{17}{256}\alpha^3a^3-\frac{7}{2048}\alpha^4a^4\right)e^4\cos 4l\right\}$$

$$+Aa\exp(-\alpha a)\frac{1-\theta^2}{1-5\theta^2}\left[\left(\frac{23}{24}+\frac{31}{24}\alpha a+\frac{29}{48}\alpha^2a^2-\frac{7}{96}\alpha^3a^3\right)e^2+\left(-\frac{527}{3072}-\frac{865}{3072}\alpha a-\frac{451}{1536}\alpha^2a^2\right.\right.$$

$$\left.\left.+\frac{2297}{4608}\alpha^3a^3-\frac{205}{9216}\alpha^4a^4-\frac{7}{1024}\alpha^5a^5\right)e^4\right]\cos 2g$$

$$+Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\left\{\left(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\right.$$

$$\times\left[\left(-\frac{11}{8}+\frac{7}{2}\alpha a+\frac{7}{8}\alpha^2a^2\right)e^2+\left(-\frac{5}{64}-\frac{13}{8}\alpha a-\frac{21}{32}\alpha^2a^2+\frac{5}{8}\alpha^3a^3+\frac{7}{64}\alpha^4a^4\right)e^4\right]\cos 2g+\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I$$

$$\left.\times\left[\left(-\frac{3}{2}+10\alpha a+\frac{7}{2}\alpha^2a^2\right)e+\left(\frac{27}{16}-14\alpha a-\frac{45}{8}\alpha^2a^2+2\alpha^3a^3+\frac{7}{16}\alpha^4a^4\right)e^3\right]\sin g\right\},$$

$$15n\frac{k_2}{a^2\eta^4}\theta^2\int(\Delta H-\Delta G)dt$$

$$=Aa\exp(-\alpha a)\frac{\theta^2(1-\theta^2)}{(1-5\theta^2)^2}\left[\left(-\frac{5}{6}-\frac{5}{12}\alpha a\right)e^2+\left(-\frac{5}{16}+\frac{5}{8}\alpha a+\frac{25}{24}\alpha^2a^2-\frac{5}{24}\alpha^3a^3-\frac{5}{144}\alpha^4a^4\right)e^4\right]\cos 2g$$

$$+Aa\exp(-\alpha a)\frac{\theta^2}{(1-5\theta^2)^2}\left\{\left(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\right[(5-5\alpha a)e^2$$

$$\left.\left.+\left(\frac{5}{8}+\frac{25}{8}\alpha a+\frac{5}{8}\alpha^2a^2-\frac{5}{8}\alpha^3a^3\right)e^4\right]\cos 2g+\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I\left[(10-10\alpha a)e+\left(-15+\frac{65}{4}\alpha a+\frac{5}{4}\alpha^2a^2-\frac{5}{4}\alpha^3a^3\right)e^3\right]\sin g\right\};$$

$$h''=h_0''+\left\{-3\frac{k_2\theta}{a^2\eta^4}+\frac{3}{8}\frac{k_2^2}{a^4\eta^8}[(-5+12\eta+9\eta^2)\theta+(-35-36\eta-5\eta^2)\theta^3]+\frac{5}{4}\frac{k_4}{a^4\eta^8}\theta(3-7\theta^2)(5-3\eta^2)\right\}nt$$

$$-3n\frac{k_2\theta}{a^2\eta^4}\int[-3\Delta L-4\Delta G+(\Delta H-\Delta G)]dt$$

$$+Aa\exp(-\alpha a)\frac{\theta}{1-5\theta^2}\left[\left(-\frac{5}{12}+\frac{1}{6}\alpha a+\frac{1}{12}\alpha^2a^2\right)e^2+\left(\frac{1}{16}-\frac{1}{8}\alpha a-\frac{5}{24}\alpha^2a^2+\frac{1}{24}\alpha^3a^3+\frac{1}{144}\alpha^4a^4\right)e^4\right]\cos 2g$$

$$+Aa\exp(-\alpha a)\frac{\theta}{1-5\theta^2}\left\{\frac{1}{12}[11+80\theta^2(1-5\theta^2)^{-1}+200\theta^4(1-5\theta^2)^{-2}]\right.$$

$$\left.-\frac{5}{18}\frac{k_4}{k_2^2}[12+640\theta^2(1-5\theta^2)^{-1}+1600\theta^4(1-5\theta^2)^{-2}]\right\}\left[\left(\frac{1}{4}-\frac{1}{4}\alpha a\right)e^2+\left(\frac{1}{32}+\frac{5}{32}\alpha a+\frac{1}{32}\alpha^2a^2-\frac{1}{32}\alpha^3a^3\right)e^4\right]\cos 2g$$

$$+Aa\exp(-\alpha a)(1-5\theta^2)^{-1}\frac{aA_{3,0}}{k_2^2}\frac{\theta}{\sin I}\left[\left(\frac{1}{6}-\frac{1}{6}\alpha a\right)e+\left(-\frac{7}{48}+\frac{13}{48}\alpha a+\frac{1}{48}\alpha^2a^2-\frac{1}{48}\alpha^3a^3\right)e^3\right]\sin g,$$

g_0'', h_0'' being constants of integration and

$$-3n\frac{k_2\theta}{a^2\eta^4}\int[-3\Delta L-4\Delta G+(\Delta H-\Delta G)]dt$$

$$=-A\frac{k_2}{a^2}a\exp(-\alpha a)\theta\left\{\left[\frac{21}{2}+\left(\frac{183}{8}+\frac{9}{2}\alpha a+\frac{21}{8}\alpha^2a^2\right)e^2+\left(\frac{4665}{128}+\frac{171}{16}\alpha a+\frac{381}{64}\alpha^2a^2+\frac{9}{16}\alpha^3a^3+\frac{21}{128}\alpha^4a^4\right)e^4\right](nt)^2\right.$$

$$+\left[(-39-21\alpha a)e+\left(-\frac{327}{4}-\frac{99}{2}\alpha a-\frac{75}{8}\alpha^2a^2-\frac{21}{8}\alpha^3a^3\right)e^3\right]\cos l+\left[\left(-\frac{171}{16}-\frac{15}{2}\alpha a-\frac{21}{16}\alpha^2a^2\right)e^2\right.$$

$$\left.+\left(-\frac{3849}{192}-\frac{245}{16}\alpha a-\frac{135}{32}\alpha^2a^2-\frac{13}{16}\alpha^3a^3-\frac{7}{64}\alpha^4a^4\right)e^4\right]\cos 2l+\left(-\frac{23}{4}-\frac{13}{3}\alpha a-\frac{9}{8}\alpha^2a^2-\frac{7}{72}\alpha^3a^3\right)e^3\cos 3l$$

$$\left.+\left(-\frac{4147}{1024}-\frac{407}{128}\alpha a-\frac{495}{512}\alpha^2a^2-\frac{17}{128}\alpha^3a^3-\frac{7}{1024}\alpha^4a^4\right)e^4\cos 4l\right\}$$

$$+Aa\exp(-\alpha a)\frac{\theta(1-\theta^2)}{(1-5\theta^2)^2}\left[\left(\frac{23}{12}+\frac{11}{4}\alpha a+\frac{31}{24}\alpha^2a^2-\frac{7}{48}\alpha^3a^3\right)e^2+\left(-\frac{431}{1536}-\frac{1057}{1536}\alpha a-\frac{611}{768}\alpha^2a^2\right.\right.$$

$$\left.\left.+\frac{2393}{2304}\alpha^3a^3-\frac{173}{4608}\alpha^4a^4-\frac{7}{512}\alpha^5a^5\right)e^4\right]\cos 2g$$

$$+Aa\exp(-\alpha a)(1-5\theta^2)^{-2}\left\{\left(\frac{1}{12}[1-11\theta^2-40\theta^4(1-5\theta^2)^{-1}]-\frac{5}{18}\frac{k_4}{k_2^2}[1-3\theta^2-8\theta^4(1-5\theta^2)^{-1}]\right)\right.$$

$$\times\left[\left(-\frac{15}{4}-8\alpha a+\frac{7}{4}\alpha^2a^2\right)e^2+\left(-\frac{9}{32}-\frac{31}{8}\alpha a-\frac{23}{16}\alpha^2a^2+\frac{11}{8}\alpha^3a^3+\frac{7}{32}\alpha^4a^4\right)e^4\right]\cos 2g+\frac{1}{6}\frac{aA_{3,0}}{k_2^2}\sin I$$

$$\left.\times\left[(-5+22\alpha a+7\alpha^2a^2)e+\left(\frac{51}{8}-\frac{125}{4}\alpha a-\frac{23}{2}\alpha^2a^2+\frac{17}{4}\alpha^3a^3+\frac{7}{8}\alpha^4a^4\right)e^3\right]\sin g\right\}.$$

These are the result of the first approximation. The second and higher approximations will give terms factored by A^{*2}/k_2, A^{*3}/k_2^2, $\cdots$ for the variables L'', G'', H'', g'', h'' and terms factored by A^{*2}/k_2^2, A^{*3}/k_2^3, $\cdots$ for long-period terms in l''. The convergence of the solution then depends on the magnitude of the ratio A^*/k_2, which is smaller than unity if effects of the drag can be assumed smaller than those of the oblateness.

The expressions for L'', G'', H'', l'', g'', h'' obtained should be substituted for the corresponding symbols in the drag-free solution (Brouwer 1959) to give the solution with drag. The six constants of integration of this solution are L_0'', G_0'', H_0'', l_0'', g_0'', h_0''.

The completely literal character of the theory makes it applicable to any actual case of satellite motion, but it is important to be aware of certain limitations that may effect the effectiveness of the theory in particular cases. In this connection the assumptions made for the earth's atmosphere are particularly relevant: (1) for the atmospheric density we have assumed the expression $\rho_0 \exp(-\alpha r)$, ρ_0 being a constant, which implies spherical symmetry of the atmosphere; (2) the rotation of the atmosphere was ignored.

Even if in specific applications these simplifying assumptions appear justified, there remain limitations introduced by series developments. For the formula for the atmospheric density we introduced into the theory a Taylor expansion

$$\exp(-x)=1-x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3\cdots,$$

in which x takes the value $-\alpha ae$ at perigee, $+\alpha ae$ at apogee. For cases with large values of αae the convergence is insufficient. In the section on Series Developments we recommended a modification available if actual numerical values for a, e are introduced. Use of the coefficients A_0, $A_1\cdots$ will improve the convergence.

Finally, the effectiveness of the theory depends on the value of A^*/k_2. For satellites with low perigee heights the large value of this ratio is unfavorable to the convergence.

The presence of k_2 in the denominator in coefficients, such as in long-period terms in the mean anomaly in the first order, raises the question what becomes of such terms in the limiting case $k_2 \rightarrow 0$; i.e., if the oblateness is neglected. It is readily seen that in this limiting case the relevant terms become linear functions of the time or constants, in view of the fact that

$$\lim_{k_2,\, k_4,\, A_{3,0}\rightarrow 0} \begin{matrix}\sin\\ \cos\end{matrix}(g,2g)=\begin{matrix}\sin\\ \cos\end{matrix}(g_0,2g_0)=\text{const.}$$

$$\lim_{k_2,\, k_4,\, A_{3,0}\rightarrow 0} \int \begin{matrix}\sin\\ \cos\end{matrix}(g,2g)dt=t\begin{matrix}\sin\\ \cos\end{matrix}(g_0,2g_0)+\text{const.}$$

Such terms may therefore be absorbed into the constants of integration.

In spite of its limitations, the theory has a considerable area of applicability. Moreover, various refinements and adaptations can be introduced later when the need for them arises. The manner in which oblateness and drag are coupled to produce mixed secular terms and long-period terms from short-period terms in the drag-free theory appears to be of considerable theoretical interest.

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A Determination of the Ellipticity of the Earth's Equator from the Motion of Two Satellites

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The effect of the most important longitude-dependent term of the geopotential on the motion of artificial satellites is expressed by simple formulae. Its numerical coefficient β can be interpreted as the ellipticity of the earth's equator; the phase constant λ_X gives the geographical longitude of the semimajor axis. Based on photoreduced Baker-Nunn observations of the satellites 1959 $\alpha 1$ and 1959 η, the values of these constants were determined to be

$$\beta=(3.21\pm0.29)\times10^{-5} \quad \text{and} \quad \lambda_X=-33^\circ\!.15\pm0^\circ\!.53.$$

INTRODUCTION

THE primary mission of satellite 1959 $\alpha 1$ (Vanguard II) was to scan the earth's cloud cover, and that of 1959 η (Vanguard III) to measure the earth's magnetic field. These experiments, however, could be conducted only for a limited period, as the transmitter of Vanguard II ceased operating 27 days after launching, and that of Vanguard III 85 days after launching. Since then the Smithsonian Astrophysical Observatory has had the responsibility for tracking both objects, and has obtained many thousands of optical observations of them by means of the Baker-Nunn camera stations and the voluntary Moonwatch teams. Because of their high quality, these observations became very valuable for different research projects, such as the study of the upper atmosphere, the determination of the earth's gravitational field, and other investigations now under way. For our present purpose, the determination of the coefficient and phase of the first longitude-dependent spherical harmonic in the geopotential, only the most accurate observations can be used. These are without question the positions obtained by the Photoreduction Section at the Cambridge headquarters of the Smithsonian Astrophysical Observatory from the Baker-Nunn photographs. For simplicity, we call these positions photoreduced observations as opposed to the field-reduced observations; the latter are the less accurate positions determined in the field, i.e., at any one of the 12 Baker-Nunn camera stations, immediately after a pass of the satellite over the station.

GEOMETRICAL INTERPRETATION OF THE GEOPOTENTIAL

Let X, Y, Z be the coordinate system of the principal axes of inertia with the origin at the center of mass, and $A\leqq B\leqq C$ the moments of inertia about these axes. Then the gravitational potential of a body can be written in the form

$$U=\frac{\mu}{r}\left\{1+\frac{A+B+C-3I}{2m_e r^2}\right\}+O\left(\frac{1}{r^4}\right), \qquad (1)$$

where $\mu=fm_e$ is the product of the constant of gravitation and the mass of the earth, and

$$I=r^{-2}(AX^2+BY^2+CZ^2)$$

is the moment of inertia about the line joining the center of mass to the point considered. It should be noted that, according to the convention in theoretical physics, we are dealing here with a force function, which is the negative of potential energy per unit mass. Expression (1), sometimes referred to as MacCullagh's formula (Jeffreys 1959), is usually expanded into a series of spherical harmonics as follows:

$$U=\frac{\mu}{r}\left\{1-J_2\left(\frac{\bar{a}_e}{r}\right)^2 P_2(\sin\Phi) - J_2^{(2)}\left(\frac{\bar{a}_e}{r}\right)^2 P_2^{(2)}(\sin\Phi)\cos 2\Lambda\right\}. \qquad (2)$$

Here Φ denotes the geocentric latitude; Λ the longitude reckoned from the principal axis X; the P's are Legendre functions; and $\bar{a}_e$ stands for the mean equatorial radius of the earth; furthermore,

$$J_2=\frac{2C-(A+B)}{2m_e\bar{a}_e^2}, \quad \text{and} \quad J_2^{(2)}=\frac{A-B}{4m_e\bar{a}_e^2}.$$

Because all gravity measurements are made with respect to a frame rotating with the earth, the geopotential Ψ is the sum of the gravitational potential U and the potential of the centrifugal force; that is,

$$\Psi=U+\tfrac{1}{2}\psi^2(X^2+Y^2), \qquad (3)$$

where ψ is the angular velocity of the earth's rotation.

To express the constants J_2 and $J_2^{(2)}$ by geometrical quantities related to the shape of the earth, and by a quantity related to ψ, we assume that the surface of the earth is very nearly an equipotential surface of the geopotential defined by Eqs. (2) and (3). It is easy to see that these equipotential surfaces for $r\approx\bar{a}_e$ are approximately ellipsoids with three unequal axes. Let $a_e>b_e>c_e$ be the axes of the equipotential ellipsoid that fits best the real surface of the earth, and let the earth's oblateness and the ellipticity of its equator be defined by the relations

$$c_e=\bar{a}_e(1-\gamma) \quad \text{and} \quad b_e=a_e(1-\beta),$$

where

$$\bar{a}_e=\tfrac{1}{2}(a_e+b_e).$$

(The above definition of the oblateness for a triaxial ellipsoid was suggested to the author by William Kaula.) Also, we introduce the small dimensionless quantity

$$\alpha=\psi^2/(\mu/\bar{a}_e^3), \tag{4}$$

which is the ratio of the squares of the earth's angular velocity and the Keplerian mean motion of a fictitious satellite with semimajor axis $a=\bar{a}_e$. It can be shown then that the quantities α, β, γ determine the coefficients in the gravitational potential as

$$J_2=\frac{2}{3}\gamma-\frac{\alpha}{3}-\frac{\gamma^2}{3}+\frac{3}{7}\alpha\gamma \tag{5}$$

and

$$J_2^{(2)}=-\tfrac{1}{6}\beta. \tag{6}$$

The value of the constant α can be determined with high accuracy. A sidereal day equals 86 164.10 sec; therefore, $\psi=7.292115\times10^{-5}$ sec^{-1}. Then, adopting from O'Keefe, Eckels, and Squires (1959) the value

$$\mu=3.98618\times10^5 \text{ km}^3 \text{ sec}^{-2}$$

and the value

$$\bar{a}_e=6.378388\times10^3 \text{ km}$$

of the international ellipsoid, we obtain from relation (4)

$$\alpha=3.46165\times10^{-3}.$$

The most accurate value of the coefficient J_2 is probably that of Kozai (1961), who has found that

$$J_2=(1.08219\pm0.00002)\times10^{-3}.$$

With these data, Eq. (5) gives for the oblateness of the earth

$$\gamma=3.35227\times10^{-3}=1/298.305.$$

The data on the ellipticity of the equator are rather contradictory in the literature. They were determined by analyzing the measures of gravity over extended areas of the earth, and seem to be influenced significantly by the distribution of the observational material. For instance, from the treatise of Heiskanen and Meinesz (1958) we list the data given in Table I, where λ_X is the geographic longitude of the direction of the principal axis X, so that for any longitude λ we have

$$\Lambda=\lambda-\lambda_X.$$

According to Jeffreys (1959), however,

$$\beta=(2.5\pm0.9)\times10^{-5}, \quad \lambda_X=0°.$$

In his textbook on celestial mechanics, Subbotin (1949) adopted the value

$$\beta=1/30000=3.3\times10^{-5}.$$

Table I.

	β	λ_X
Helmert, 1915	3.6×10^{-5}	$-17°$
Heiskanen, 1924	5.4×10^{-5}	$+18°$
Heiskanen, 1928	3.8×10^{-5}	$0°$
Niskanen, 1945	4.6×10^{-5}	$-4°$
Uotila, 1957	2.1×10^{-5}	$-6°$

In his detailed study, Jongolovich (1957) derived the equivalent of the values

$$\beta=3.57\times10^{-5}, \quad \lambda_X=-7\overset{\circ}{.}7.$$

From a recent paper by Kaula (1959) one can infer the values

$$\beta=3.7\times10^{-6} \quad \text{and} \quad \lambda_X=-20\overset{\circ}{.}8.$$

EFFECT OF THE ELLIPTICITY OF THE EARTH'S EQUATOR ON THE MOTION OF SATELLITES

Several investigators have pointed out that the ellipticity of the earth's equator might have a detectable effect on the motion of artificial satellites, and therefore a careful analysis of accurate satellite observations could lend itself to a better determination of the constants β and λ_X. In this connection let us refer to the papers by O'Keefe and Batchlor (1957), Robe (1959), Cook (1960), Sehnal (1960), and Musen (1960).

To derive the perturbations caused by the presence of the sectorial harmonic in question, we can confine ourselves to a very simple theory. The disturbing function, which is

$$R=\frac{\beta}{2}\frac{\mu}{a_e}\left(\frac{a_e}{r}\right)^3 \cos^2\Phi \cos2\Lambda,$$

must first be transformed into a function of the orbital elements a or n, e, I, M, ω, and Ω of the satellite. Let

v = the true anomaly,
$u=v+\omega$ the argument of latitude,
w = the projection of u on the equator, that is, the longitude of the satellite reckoned from the ascending node of its orbit,
θ_G and θ_X = the mean sidereal time at Greenwich and at the intersection of the principal axis X with the surface of the earth.

Note that

$$\Lambda=w-(\theta_X-\Omega),$$

and

$$\lambda_X=\theta_X-\theta_G.$$

The angles Φ, w are related to the angles I, u by the formulae

$$\cos\Phi \cos w=\cos u$$

and

$$\cos\Phi \sin w=\cos I \sin u,$$

in accordance with spherical trigonometry. These equations can be written in the complex form

$$\cos\Phi \exp(iw) = \cos u + i \cos I \sin u = \cos^2\tfrac{1}{2}I \exp(iu) + \sin^2\tfrac{1}{2}I \exp(-iu),$$

which immediately yields

$$\cos\Phi \exp(i\Lambda) = \cos^2\tfrac{1}{2}I \exp[i(u-\theta_X+\Omega)] + \sin^2\tfrac{1}{2}I \exp[-i(u+\theta_X-\Omega)].$$

But

$$\cos^2\Phi \cos 2\Lambda = \mathrm{Re}\{[\cos\Phi \exp(i\Lambda)]^2\},$$

so that introducing the abbreviations

$$c = \cos I, \quad s = \sin I, \quad n^2 = \mu/a^3,$$

we obtain the disturbing function in the form

$$R = \tfrac{1}{8}\beta a_e^2 n^2 (a/r)^3 \{2s^2 \cos 2(\theta_X-\Omega) + (1+c)^2 \cos 2(u-\theta_X+\Omega) + (1-c)^2 \cos 2(u+\theta_X-\Omega)\}.$$

General formulae for the disturbing function of any tesseral harmonic can be found in a paper by Groves (1960).

In our problem it is not necessary to carry out the development of this function into a Fourier series in terms of the mean anomaly M, because the contribution of the short-periodic terms to the perturbations turns out to be negligibly small. As the Fourier series of the functions

$$(a/r)^3 \cos 2v \quad \text{and} \quad (a/r)^3 \sin 2v$$

do not contain constant terms, and the constant term in $(a/r)^3$ is $(1-e^2)^{-\frac{3}{2}}$, the "long-periodic" part of R becomes simply

$$\bar{R} = (\tfrac{1}{4}\beta) a_e^2 n^2 (1-e^2)^{-\frac{3}{2}} s^2 \cos 2(\theta_X-\Omega). \tag{7}$$

The perturbations caused by the oblateness of the earth concern us here only to the extent that Ω is not constant, but rather a linear function of time:

$$\Omega = \Omega_0 + \Omega_1(t-t_0),$$

so that

$$\theta_X - \Omega = \theta_{X0} - \Omega_0 + (\theta_1-\Omega_1)(t-t_0):$$

Substituting function (7) into the differential equations of the orbital elements [see, for example, Moulton (1959)] and integrating with respect to time, we obtain at once

$$\begin{aligned} \delta\omega &= \beta \frac{n a_e^2 (3-5c^2)}{8(\theta_1-\Omega_1)p^2} \sin 2(\theta_X-\Omega), \\ \delta\Omega &= \beta \frac{n a_e^2 c}{4(\theta_1-\Omega_1)p^2} \sin 2(\theta_X-\Omega), \\ \delta I &= \beta \frac{n a_e^2 s}{4(\theta_1-\Omega_1)p^2} \cos 2(\theta_X-\Omega), \\ \delta M &= \beta \frac{3 n a_e^2 (1-e^2)^{\frac{1}{2}} s^2}{8(\theta_1-\Omega_1)p^2} \sin 2(\theta_X-\Omega). \end{aligned} \tag{8}$$

The semimajor axis a and the eccentricity e do not undergo long-periodic perturbations due to the ellipticity of the equator. The question now is how to use these formulae for the determination of the constants β and λ_X, and whether the accuracy of the observations made with the Baker-Nunn cameras is sufficient for such a determination.

COMPUTATION OF THE CONSTANTS β AND λ_X

In the Differential Orbit Improvement Program of the Smithsonian Astrophysical Observatory developed for the IBM 704 computer by G. Veis and Ch. H. Moore, the orbital elements ω, Ω, I, e, M can be represented as polynomials up to the seventh degree, plus sine terms and exponential functions of the time t. As many as 23 of the data figuring in such a representation can be varied in an iterative least-squares solution, but we usually limit ourselves to a much smaller number of unknowns in order to obtain meaningful results. It takes some experience to decide properly which quantities should be varied in a particular application.

The numerical material for the present investigation was obtained in December, 1960 by several consecutive runs of this program, as applied to 187 photoreduced Baker-Nunn observations of satellite 1959 $\alpha 1$ covering the interval April 6 through 25, 1960; and to 216 observations of satellite 1959 η covering the interval May 8 through 27, 1960. The use of two satellites is not a theoretical necessity, but a precautionary measure in order to give two independent determinations of the constants in question. While it is not possible here to furnish all the details of these orbit computations, their important features can be summarized as follows. Because the phase at which the trigonometric terms (8) are "observed" at a particular station varies only slightly from day to day, it is desirable to use a long interval of time for the computation of the constants β and λ_X. However, even in the case of such relatively high orbiting satellites as 1959 $\alpha 1$ and 1959 η, the effect of the air drag cannot be described with sufficient accuracy in too long an interval of time by simple polynomial expressions. As a compromise, some numerical experimentation led to the 20-day intervals noted above, with third-degree polynomials in the mean anomaly M. The best possible polynomial representation of all the orbital elements was determined, treating ω and Ω as polynomials of the second degree, I and e as polynomials of the first degree, and M as a polynomial of the third degree. This means that the long-range direct and indirect effects of the air drag, of the radiation pressure, of secular and long-periodic perturbations due to the even and odd zonal harmonics in the earth's gravitational potential, and of luni-solar perturbations were treated in an empirical way. Analytical expressions for the first-order short-periodic perturbations due to the oblateness of the earth are incorporated in the computer program. The coefficients

TABLE II. Results of computations.

	Satellite	
	1959 $\alpha 1$	1959 η
t_0	April 16.0, 1960	May 19.0, 1960
θ_{G0}	$204^{\circ}.1383$	$236^{\circ}.6646$
θ_1	1.002738 rev/day	
Ω_0	$138^{\circ}.0499$	$155^{\circ}.1312$
Ω_1	$-3^{\circ}.51067$/day	$-3^{\circ}.27567$/day
n	11.463937 rev/day	11.069014 rev/day
a	1.302343 a_e	1.333153 a_e
e	0.1646379	0.1891083
$q=a(1-e)-a_e$	561 km	517 km
I	$32^{\circ}.8796$	$33^{\circ}.3569$

of these polynomials were improved by including the trigonometric terms (8) into the orbital elements with the constants

$$\beta=3.8\times10^{-5} \quad \text{and} \quad \lambda_X=0^{\circ}$$

of Heiskanen. In several consecutive runs these constants were then treated as unknowns only in the right ascension of the node Ω, because one would expect no short-periodic air drag effects in this orbital element, and because the amplitude of $\delta\Omega$ turned out to be the biggest one. The improved values of β and λ_X were substituted into $\delta\omega$, $\delta\Omega$, δI, and δM, and then the computation of these constants was repeated until the new values agreed with the old ones within their standard errors. The relevant results of these computations are given in Table II. Consequently, expressions (8) become

$$\begin{array}{lll}
\delta\omega & -\beta\times0.4641\sin2(\theta_X-\Omega) & -\beta\times0.4041\sin2(\theta_X-\Omega) \\
\delta\Omega & \beta\times1.4808\sin2(\theta_X-\Omega) & \beta\times1.3824\sin2(\theta_X-\Omega) \\
\delta I & \beta\times0.9572\cos2(\theta_X-\Omega) & \beta\times0.9100\cos2(\theta_X-\Omega) \\
\delta M & \beta\times0.7688\sin2(\theta_X-\Omega) & \beta\times0.7370\sin2(\theta_X-\Omega).
\end{array}$$

In the course of the trigonometric least-squares fit to the observations, the IBM 704 electronic computer actually obtained the amplitude of $\delta\Omega$ and twice the phase angle $\theta_{X0}-\Omega_0=\lambda_X+(\theta_{G0}-\Omega_0)$. From these data we immediately obtain

$$\begin{array}{lll}
\beta & (3.47\pm0.28)\times10^{-5} & (2.88\pm0.32)\times10^{-5} \\
\lambda_X & -33^{\circ}.5\pm1^{\circ}.9 & -32^{\circ}.3\pm3^{\circ}.1.
\end{array}$$

The agreement between the two sets of data seems reasonable; taking the weighted mean values, we obtain

$$\beta=(3.21\pm0.29)\times10^{-5},$$

and

$$\lambda_X=-33^{\circ}.15\pm0^{\circ}.53.$$

This numerical value of the ellipticity gives the perturbation $\delta\omega, \cdots$ expressed in radians. For other angular units we note that

$$\beta=5.11\times10^{-6}\ \text{rev}=1^{\circ}.84\times10^{-3}=6''.62.$$

According to the definition of the quantity β, the difference between the axes a_e and b_e is 205 m.

Needless to say, this determination of the coefficients of the second sectorial harmonic is far from being a final one. It will be improved as soon as more accurate coordinates of the Baker-Nunn camera stations are available. Also, the effect of other tesseral harmonics on this determination should be examined, possibly by using satellites with essentially different orbital elements.

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Departures from the Russell Model in TX Ursae Majoris

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Yellow, blue, and violet photoelectric light curves have been obtained for TX UMa. Additional observations place the variable on the *U*, *B*, *V* system. It is demonstrated that the light curve is variable, at least at certain phases. The outside-eclipse variations exhibit departures from the Russell model as a function of wavelength of observation. Consistency in the solution of primary eclipse can be obtained if one abandons the Russell model for an ordinary body eclipse in favor of one comprising two sharp-edged stars, one of which is surrounded by an extensive but sharp-edged atmosphere.

I. INTRODUCTION

THE most recent photometric studies of TX UMa are those by Wood (1946) and Huffer and Eggen (1947). Each of these articles adequately summarizes the past history of the eclipsing pair. Plavec (1960) has collected the published times of minimum light and studied the apsidal motion of the orbit. In addition, Struve and Huang (1956) have considered the possibility that the hot component is at times surrounded by an emitting ring composed of material ejected from the cool star.

The notation used in the present paper for characteristics explicable by the Russell model is that recommended by Russell and Merrill (1952a).

II. OBSERVATIONS

In 1956 and 1957 TX UMa was observed with the photoelectric photometers on the 36-inch and 12.5-inch reflectors of the Steward Observatory. A total of 372 yellow, 365 blue, and 368 ultraviolet (hereafter, violet) observations were obtained. Those observations taken with the smaller reflector have been transformed to the instrumental system of the 36-inch telescope by the technique described by Koch (1960). All observations have been corrected for differential extinction using the mean coefficients given in the article just cited. The comparison star is BD +47°1797, and it was checked at least once in each color every night against BD +47°1799. The check star is ADS 7886, seemingly an optical double, and both components have been observed through the aperture of the diaphragm slide. Study of the magnitude differences between the comparison and check stars indicates that there are no measurable night or seasonal variations in either star. The observations are listed in Table I (a, b, and c) in which time is counted from Greenwich Mean Noon. The yellow observations are collected into one light curve in Fig. 1.

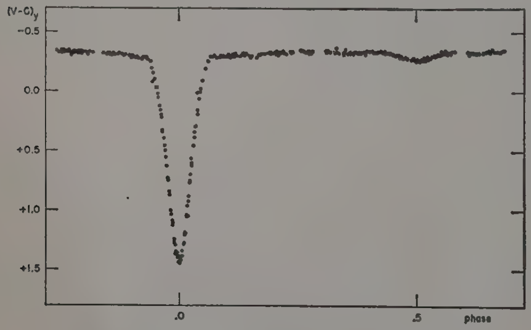

FIG. 1. Yellow observations of TX UMa collected onto one light curve.

On two nights the comparison and check stars were transformed to the *U,B,V* system of Johnson and Morgan (1953) by using δ UMa and Groombridge 1618 as standards. The magnitudes and color indices for the comparison stars, determined by least-squares solutions, are shown in Table II. This table also lists the probable error for each determination and the number of observations of these stars. From these observations it is possible to scale the light variation of TX UMa on the *U,B,V* system, and this is shown in the last four lines of Table II.

III. EPOCH AND PERIOD

Newly determined heliocentric times of minimum light in primary eclipse are JD 2435541.8634 and JD 2435584.7500. These minima, together with all others published and available, have been tested against the ephemeris given by Wood. It is concluded that his light elements can be slightly improved to:

$$\text{Pr. Min.} = 2416426.781 + 3.0633175\,E + \\ +0.021\sin(0^{\circ}\!.09474\,E + 64^{\circ}\!.42).$$

With these elements primary eclipse is symmetric. Since the orbital eccentricity is small, this is expected. Plavec's ephemeris differs slightly from this one and shifts primary minimum from zero phase. Accordingly, the phases of the observations in Table I have been computed with the elements given above. The exact behavior of secondary eclipse in the new light curves is dealt with in a subsequent section.

IV. VARIATIONS OUTSIDE ECLIPSES

Night-by-night plots of the observations demonstrated that there are no consistent seasonal differences

Table Ia. Yellow observations of TX UMa.

V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase
$-0^{\mathrm{m}}351$	528.8459	0.7507	$-0^{\mathrm{m}}273$	547.7884	0.9344	$-0^{\mathrm{m}}283$	555.6311	0.4945
−0.345	.8581	0.7547	+0.049	.8561	0.9565	−0.272	.6384	0.4969
−0.331	.8667	0.7575	+0.121	.8651	0.9594	−0.259	.6488	0.5003
−0.337	.8754	0.7604	+0.191	.8728	0.9619	−0.275	.6557	0.5026
−0.340	.8845	0.7633	+0.325	.8835	0.9654	−0.273	.6651	0.5056
−0.355	530.6851	0.3511	+0.392	.8912	0.9679	−0.260	.6721	0.5079
−0.324	.6987	0.3556	+0.495	.9009	0.9711	−0.274	.6814	0.5110
−0.334	.7105	0.3594	+0.623	.9085	0.9736	−0.264	.6894	0.5136
−0.325	.7198	0.3625	+0.745	.9193	0.9771	−0.282	.6998	0.5170
−0.329	.7285	0.3653	+0.868	.9269	0.9796	−0.283	.7071	0.5194
−0.331	.7372	0.3681	+1.034	.9380	0.9832	−0.286	.7179	0.5229
−0.335	.7459	0.3710	+1.123	.9467	0.9860	−0.277	.7262	0.5256
−0.334	.7556	0.3742	+1.273	.9582	0.9898	−0.288	.7370	0.5291
−0.325	.7684	0.3783	+1.348	.9679	0.9930	−0.301	.7519	0.5340
−0.333	.7771	0.3812	+1.395	.9801	0.9969	−0.311	.7596	0.5365
−0.331	.7865	0.3842	+1.447	.9901	0.0002	−0.299	.7693	0.5397
−0.337	.7955	0.3872	−0.331	551.6677	0.1945	−0.311	.7790	0.5428
−0.336	.8063	0.3907	−0.341	.6861	0.2067	−0.316	.7887	0.5460
−0.319	.8150	0.3935	−0.310	.6959	0.2099	−0.314	.7978	0.5490
−0.345	.8247	0.3967	−0.326	.7084	0.2140	−0.344	.8096	0.5528
−0.345	.8337	0.3996	−0.347	.7170	0.2168	−0.333	.8169	0.5552
−0.343	.8441	0.4030	−0.343	.7282	0.2205	−0.326	.8273	0.5586
−0.343	.8577	0.4075	−0.326	.7354	0.2228	−0.327	.8346	0.5613
−0.336	.8674	0.4106	−0.320	.7462	0.2263	−0.345	.8443	0.5641
−0.333	.8778	0.4140	−0.340	.7743	0.2355	−0.339	.8519	0.5666
−0.333	.8917	0.4186	−0.329	.7809	0.2377	−0.333	.8620	0.5699
−0.336	.9014	0.4217	−0.335	.7900	0.2406	−0.335	.8693	0.5723
−0.341	.9136	0.4257	−0.349	.8386	0.2565	−0.339	.8790	0.5755
−0.334	.9278	0.4304	−0.354	.8479	0.2595	−0.341	.8863	0.5779
−0.340	.9389	0.4340	−0.345	.8552	0.2619	−0.339	.8974	0.5815
−0.330	.9490	0.4373	−0.279	552.6201	0.5116	−0.311	556.7164	0.8488
−0.326	.9584	0.4404	−0.271	.6347	0.5164	−0.313	.7310	0.8536
−0.327	.9702	0.4442	−0.269	.6417	0.5187	−0.317	.7383	0.8560
+1.187	538.8366	0.0121	−0.274	.6510	0.5217	−0.319	.7477	0.8591
+1.040	.8495	0.0163	−0.292	.6597	0.5245	−0.322	.7550	0.8614
+0.880	.8613	0.0202	−0.286	.6698	0.5278	−0.313	.7664	0.8652
+0.766	.8696	0.0229	−0.284	.6771	0.5302	−0.307	.7748	0.8679
+0.576	.8839	0.0276	−0.297	.6861	0.5332	−0.316	.7855	0.8714
+0.444	.8953	0.0313	−0.304	.6927	0.5353	−0.322	.7946	0.8744
+0.299	.9068	0.0350	−0.310	.7021	0.5384	−0.319	.8039	0.8774
+0.010	.9179	0.0387	−0.298	.7090	0.5406	−0.315	.8116	0.8799
−0.023	.9412	0.0463	−0.306	.7156	0.5428	−0.308	.8241	0.8840
−0.093	.9523	0.0499	−0.318	.7257	0.5461	−0.315	.8328	0.8868
−0.195	.9738	0.0569	−0.322	.7385	0.5503	−0.320	557.7720	0.1934
−0.186	541.6931	0.9446	−0.320	.7496	0.5539	−0.322	.7858	0.1979
+0.152	.7425	0.9607	−0.319	.7566	0.5562	−0.323	.7928	0.2002
+0.215	.7491	0.9629	−0.322	.7667	0.5595	−0.326	.8015	0.2031
+0.334	.7591	0.9662	−0.324	.7739	0.5618	−0.318	.8091	0.2055
+0.456	.7702	0.9698	−0.318	.7837	0.5650	−0.325	559.7541	0.8405
+0.626	.7831	0.9740	−0.340	.7910	0.5674	−0.310	.7684	0.8451
+1.088	.8181	0.9854	−0.326	.8007	0.5706	−0.315	.7753	0.8474
+1.255	.8293	0.9891	−0.324	.8076	0.5728	−0.326	.7854	0.8507
+1.364	.8376	0.9918	−0.289	554.6200	0.1645	−0.327	.7934	0.8533
+1.392	.8497	0.9957	−0.312	.6329	0.1689	−0.313	.8027	0.8563
+1.435	.8584	0.9986	−0.310	.6395	0.1708	−0.309	.8093	0.8585
+1.419	.8727	0.0032	−0.321	.6488	0.1739	−0.257	569.6205	0.0612
+1.386	.8810	0.0059	−0.325	.6544	0.1757	−0.295	.6365	0.0665
+1.272	.8931	0.0099	−0.320	.6655	0.1793	−0.294	.6435	0.0688
+1.192	.9015	0.0126	−0.320	.6721	0.1815	−0.291	.6525	0.0717
+1.043	.9133	0.0165	−0.326	.6839	0.1853	−0.300	.6612	0.0745
+0.924	.9220	0.0193	−0.325	.6902	0.1874	−0.287	.6705	0.0776
+0.745	.9341	0.0233	−0.327	.6992	0.1903	−0.283	.6771	0.0797
+0.605	.9435	0.0263	−0.323	.7058	0.1925	−0.302	.7063	0.0893
−0.349	542.8145	0.3107	−0.325	.7159	0.1958	−0.302	.7171	0.0928
−0.350	.8244	0.3139	−0.334	.7291	0.2001	−0.299	.7240	0.0950
−0.356	543.8844	0.6599	−0.333	.7388	0.2033	−0.292	.7334	0.0981
−0.341	.8972	0.6641	−0.329	.7495	0.2068	−0.297	.7417	0.1008
−0.338	.9083	0.6677	−0.322	.7565	0.2090	−0.300	.7521	0.1042
−0.340	.9160	0.6702	−0.337	.7662	0.2122	−0.294	.7876	0.1158
−0.348	.9271	0.6739	−0.334	.7732	0.2145	−0.299	.7966	0.1187
−0.337	.9347	0.6763	−0.335	.7846	0.2182	−0.338	579.6431	0.3331
−0.347	.9448	0.6796	−0.337	.7923	0.2207	−0.327	.6588	0.3382
−0.351	.9552	0.6830	−0.276	555.6175	0.4901	−0.329	.6865	0.3473

TABLE Ia.—*Continued.*

V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase
$-0\overset{\mathrm{m}}{.}342$	579.6970	0.3507	$-0\overset{\mathrm{m}}{.}331$	818.9257	0.4449	$-0\overset{\mathrm{m}}{.}316$	860.8849	0.1421
+0.336	584.6440	0.9656	−0.326	.9330	0.4472	−0.339	862.7871	0.7630
+0.550	.6645	0.9723	−0.319	.9424	0.4503	−0.335	.7957	0.7659
+0.728	.6782	0.9767	−0.314	.9493	0.4526	−0.335	.8044	0.7687
+0.841	.6865	0.9794	−0.306	.9663	0.4581	−0.325	.8131	0.7715
+1.002	.6969	0.9828	−0.313	.9730	0.4603	−0.318	.8218	0.7744
+1.108	.7059	0.9858	−0.309	.9830	0.4636	−0.306	893.8262	0.8955
+1.240	.7178	0.9897	−0.296	.9903	0.4659	−0.286	.8404	0.9001
+1.289	.7254	0.9921	−0.302	819.0010	0.4694	−0.291	.8491	0.9030
+1.354	.7368	0.9958	−0.286	.0104	0.4725	−0.291	.8578	0.9058
+1.389	.7441	0.9982	−0.291	.0278	0.4782	−0.288	.8668	0.9087
+1.389	.7563	0.0022	−0.282	.0361	0.4809	−0.293	.8758	0.9117
+1.340	.7671	0.0058	−0.334	.8931	0.7607	−0.284	.8848	0.9146
+1.285	.7778	0.0093	−0.330	.9035	0.7641	−0.276	.9057	0.9214
+1.207	.7848	0.0115	−0.341	.9119	0.7668	−0.289	.9157	0.9247
+1.052	.7955	0.0150	−0.309	.9296	0.7726	−0.288	.9251	0.9278
+0.961	.8039	0.0178	−0.321	.9362	0.7747	−0.287	.9345	0.9308
+0.698	.8223	0.0238	−0.328	.9456	0.7778	−0.289	.9446	0.9341
+0.564	.8316	0.0268	−0.326	.9522	0.7800	−0.268	.9533	0.9370
+0.452	.8428	0.0305	−0.328	.9619	0.7831	−0.238	.9626	0.9400
+0.339	.8507	0.0331	−0.326	.9685	0.7853	−0.199	.9713	0.9429
+0.155	.8688	0.0390	−0.322	.9778	0.7883	−0.157	.9803	0.9458
+0.066	.8771	0.0417	−0.325	.9844	0.7905	−0.104	.9893	0.9487
+0.003	.8879	0.0452	−0.320	.9942	0.7937	−0.037	.9991	0.9519
−0.334	588.7685	0.3120	−0.295	820.9161	0.0946	+0.018	894.0084	0.9550
−0.336	.7831	0.3167	−0.312	.9484	0.1052	−0.336	931.7218	0.2662
−0.343	.7911	0.3193	−0.290	.9595	0.1088	−0.335	.7357	0.2707
−0.321	.8025	0.3231	−0.308	.9672	0.1113	−0.342	.7444	0.2735
−0.341	.8105	0.3257	−0.315	.9793	0.1153	−0.329	.7530	0.2764
−0.330	.8213	0.3292	−0.321	.9925	0.1196	−0.330	.7617	0.2792
−0.369	.8313	0.3325	−0.302	821.0026	0.1229	−0.343	.7704	0.2820
−0.355	589.6695	0.6061	−0.336	837.8065	0.6083	−0.312	946.6444	0.1375
−0.290	615.6486	0.0868	−0.319	840.8785	0.6112	−0.316	.6604	0.1427
−0.296	.6643	0.0919	−0.330	.8875	0.6141	−0.309	.6698	0.1458
−0.290	.6733	0.0948	−0.327	.8945	0.6164	−0.306	.6784	0.1486
−0.346	617.6554	0.7419	−0.330	.9035	0.6193	−0.289	.6874	0.1516
−0.328	.6714	0.7471	−0.330	.9098	0.6214	−0.301	.6965	0.1545
−0.325	.6790	0.7496	−0.327	.9184	0.6242	−0.342	954.7295	0.7769
−0.328	.6887	0.7528	−0.328	.9250	0.6263	−0.339	.7420	0.7809
−0.349	.6981	0.7558	−0.328	.9341	0.6293	−0.345	.7514	0.7840
−0.336	.7342	0.7676	−0.330	.9410	0.6316	−0.333	.7611	0.7872
−0.325	.7443	0.7709	−0.331	.9695	0.6409	−0.336	.7702	0.7901
−0.331	.7537	0.7740	−0.319	.9792	0.6440	−0.339	.7795	0.7932
+0.626	624.6535	0.0263	−0.338	.9862	0.6463	−0.330	.7886	0.7961
+0.388	.6719	0.0323	−0.346	.9959	0.6495	−0.335	.7973	0.7990
+0.290	.6823	0.0357	−0.349	841.0032	0.6519	−0.331	.8059	0.8018
+0.191	.6931	0.0393	−0.337	.0132	0.6551	−0.331	.8164	0.8052
+0.063	.7028	0.0424	−0.301	860.8446	0.1289	−0.315	.8250	0.8080
−0.325	818.8958	0.4351	−0.311	.8561	0.1327	−0.335	.8351	0.8113
−0.329	.9097	0.4396	−0.326	.8655	0.1358	−0.332	.8441	0.8143
−0.334	.9163	0.4418	−0.304	.8731	0.1382	−0.330	.8532	0.8172

within the light curves. On JD 2435818–9 and JD 2435893–4, however, the observations are respectively systematically high and low in a suspicious manner. Nonetheless, it was possible to analyze the observations outside eclipse for the expected Fourier expansions according to the graphical method of Russell and Merrill (1952b). These are given as

$$\mathcal{L}(y)=0.9735-0.0192\cos\theta-0.0139\cos 2\theta,$$

$$\mathcal{L}(b)=0.9819-0.0139\cos\theta-0.0056\cos 2\theta,$$

$$\mathcal{L}(u)=0.9780-0.0162\cos\theta-0.0058\cos 2\theta.$$

The probable residuals for a single observation from these computed variations are $\pm 0\overset{\mathrm{m}}{.}006$ in yellow and blue and $\pm 0\overset{\mathrm{m}}{.}008$ in violet. Terms in $\cos\theta$ higher than those listed and all terms in $\sin\theta$ are small enough to be safely neglected.

V. PRELIMINARY RECTIFICATIONS AND SOLUTIONS

From the solutions given previously by O'Keefe (1938) and Wood it is possible to derive approximate values of the inclination of the orbit to the plane of the sky and the mean ratio of the surface brightnesses of the components as a function of wavelength. With these quantities known, the observations were provisionally rectified for reflection and the mean ellipticity of figure of the stars according to the following equations:

TABLE Ib. Blue observations of TX UMa.

V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase
$-0\overset{\mathrm{m}}{.}514$	528.8480	0.7514	$+0\overset{\mathrm{m}}{.}007$	547.8658	0.9596	$-0\overset{\mathrm{m}}{.}430$	555.6478	0.5000
−0.508	.8574	0.7545	+0.095	.8735	0.9621	−0.451	.6547	0.5022
−0.494	.8660	0.7573	+0.229	.8842	0.9656	−0.442	.6641	0.5053
−0.487	.8747	0.7601	+0.318	.8919	0.9681	−0.446	.6710	0.5076
−0.497	.8838	0.7631	+0.459	.9016	0.9713	−0.441	.6804	0.5106
−0.514	530.6844	0.3509	+0.591	.9092	0.9738	−0.440	.6884	0.5133
−0.484	.6980	0.3554	+0.766	.9200	0.9773	−0.456	.6988	0.5166
−0.488	.7098	0.3592	+0.909	.9276	0.9798	−0.439	.7061	0.5190
−0.486	.7191	0.3622	+1.117	.9387	0.9834	−0.451	.7169	0.5226
−0.494	.7278	0.3651	+1.249	.9474	0.9863	−0.450	.7252	0.5253
−0.497	.7365	0.3679	+1.429	.9589	0.9900	−0.453	.7360	0.5288
−0.498	.7410	0.3694	+1.557	.9686	0.9932	−0.459	.7509	0.5337
−0.495	.7549	0.3739	+1.640	.9808	0.9972	−0.473	.7585	0.5361
−0.489	.7678	0.3781	+1.661	.9908	0.0004	−0.468	.7682	0.5393
−0.483	.7764	0.3809	−0.470	551.6670	0.2005	−0.473	.7780	0.5425
−0.490	.7858	0.3840	−0.495	.6854	0.2065	−0.472	.7877	0.5457
−0.489	.7948	0.3869	−0.461	.6952	0.2097	−0.466	.7946	0.5479
−0.498	.8056	0.3905	−0.491	.7098	0.2145	−0.500	.8085	0.5526
−0.489	.8143	0.3933	−0.490	.7164	0.2166	−0.485	.8158	0.5548
−0.500	.8240	0.3965	−0.483	.7275	0.2202	−0.470	.8262	0.5582
−0.500	.8330	0.3994	−0.485	.7348	0.2226	−0.478	.8335	0.5606
−0.514	.8434	0.4028	−0.483	.7455	0.2261	−0.479	.8432	0.5638
−0.510	.8570	0.4073	−0.481	.7736	0.2353	−0.492	.8509	0.5663
−0.483	.8667	0.4104	−0.480	.7802	0.2374	−0.496	.8606	0.5695
−0.488	.8771	0.4138	−0.476	.7893	0.2404	−0.501	.8682	0.5719
−0.473	.8875	0.4172	−0.494	.8379	0.2563	−0.493	.8780	0.5751
−0.493	.9007	0.4215	−0.490	.8473	0.2593	−0.498	.8853	0.5775
−0.494	.9129	0.4255	−0.487	.8545	0.2617	−0.483	.8964	0.5812
−0.498	.9271	0.4301	−0.456	552.6187	0.5112	−0.476	556.7154	0.8485
−0.498	.9382	0.4338	−0.461	.6337	0.5161	−0.477	.7300	0.8533
−0.480	.9483	0.4371	−0.457	.6406	0.5183	−0.473	.7373	0.8557
−0.485	.9577	0.4401	−0.452	.6500	0.5214	−0.470	.7467	0.8587
−0.492	.9695	0.4440	−0.464	.6618	0.5252	−0.478	.7539	0.8611
+1.326	538.8373	0.0124	−0.461	.6687	0.5275	−0.472	.7633	0.8641
+1.112	.8502	0.0166	−0.456	.6760	0.5299	−0.464	.7730	0.8673
+0.887	.8620	0.0204	−0.467	.6851	0.5328	−0.476	.7845	0.8711
+0.755	.8703	0.0231	−0.475	.6917	0.5350	−0.473	.7935	0.8740
+0.517	.8846	0.0278	−0.467	.7010	0.5380	−0.478	.8029	0.8771
+0.356	.8960	0.0315	−0.468	.7080	0.5403	−0.475	.8105	0.8796
+0.185	.9075	0.0353	−0.471	.7146	0.5425	−0.474	.8230	0.8836
+0.003	.9186	0.0389	−0.472	.7246	0.5457	−0.477	.8317	0.8865
−0.179	.9418	0.0465	−0.492	.7375	0.5499	−0.470	557.7695	0.1926
−0.235	.9516	0.0497	−0.480	.7486	0.5536	−0.474	.7848	0.1976
−0.354	.9731	0.0567	−0.484	.7555	0.5558	−0.477	.7917	0.1999
+0.025	541.7418	0.9605	−0.480	.7656	0.5591	−0.471	.8004	0.2027
+0.123	.7497	0.9631	−0.480	.7729	0.5615	−0.468	.8081	0.2052
+0.245	.7598	0.9664	−0.471	.7826	0.5647	−0.478	559.7531	0.8402
+0.402	.7709	0.9700	−0.481	.7899	0.5670	−0.472	.7673	0.8448
+0.609	.7838	0.9742	−0.474	.7996	0.5702	−0.481	.7743	0.8471
+1.230	.8188	0.9856	−0.485	.8066	0.5725	−0.477	.7843	0.8503
+1.409	.8300	0.9893	−0.467	554.6190	0.1642	−0.483	.7923	0.8529
+1.510	.8383	0.9920	−0.463	.6318	0.1683	−0.477	.8017	0.8560
+1.643	.8504	0.9960	−0.472	.6384	0.1705	−0.480	.8083	0.8582
+1.686	.8591	0.9988	−0.472	.6478	0.1736	−0.472	569.6355	0.0662
+1.651	.8734	0.0034	−0.477	.6544	0.1757	−0.467	.6424	0.0684
+1.609	.8817	0.0062	−0.480	.6645	0.1790	−0.453	.6515	0.0714
+1.445	.8938	0.0101	−0.480	.6711	0.1812	−0.477	.7053	0.0889
+1.314	.9022	0.0128	−0.479	.6801	0.1841	−0.465	.7160	0.0924
+1.122	.9140	0.0167	−0.485	.6891	0.1870	−0.464	.7230	0.0947
+0.957	.9227	0.0195	−0.482	.6982	0.1900	−0.463	.7324	0.0978
+0.728	.9348	0.0235	−0.483	.7048	0.1922	−0.459	.7407	0.1004
+0.551	.9442	0.0266	−0.488	.7148	0.1954	−0.465	.7511	0.1039
−0.488	542.8138	0.3104	−0.476	.7280	0.1997	−0.461	.7865	0.1144
−0.505	.8237	0.3137	−0.476	.7377	0.2029	−0.474	.7938	0.1178
−0.498	543.8837	0.6597	−0.470	.7485	0.2064	−0.482	579.6421	0.3328
−0.489	.8965	0.6639	−0.475	.7554	0.2087	−0.489	.6577	0.3379
−0.478	.9077	0.6675	−0.487	.7652	0.2119	−0.493	.6855	0.3469
−0.479	.9153	0.6700	−0.487	.7721	0.2141	−0.495	.6959	0.3503
−0.498	.9264	0.6736	−0.481	.7836	0.2179	+0.231	584.6443	0.9657
−0.482	.9340	0.6761	−0.486	.7912	0.2204	+0.456	.6613	0.9712
−0.483	.9441	0.6794	−0.467	555.6165	0.4898	+0.764	.6813	0.9777
−0.473	.9511	0.6817	−0.455	.6300	0.4942	+1.073	.6976	0.9831
−0.065	547.8575	0.9569	−0.444	.6373	0.4966	+1.222	.7066	0.9860

TABLE Ib.—*Continued.*

V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase
$+1\overset{m}{.}406$	584.7184	0.9899	$-0\overset{m}{.}487$	818.9737	0.4605	$-0\overset{m}{.}494$	862.8051	0.7689
+1.487	.7261	0.9924	−0.487	.9837	0.4638	−0.479	.8138	0.7718
+1.604	.7375	0.9961	−0.463	.9910	0.4662	−0.486	.8225	0.7746
+1.639	.7448	0.9985	−0.477	819.0017	0.4697	−0.459	893.8269	0.8957
+1.657	.7591	0.0032	−0.474	.0111	0.4727	−0.449	.8411	0.9003
+1.560	.7678	0.0060	−0.463	.0285	0.4784	−0.449	.8498	0.9032
+1.449	.7785	0.0095	−0.461	.0368	0.4812	−0.453	.8584	0.9060
+1.343	.7855	0.0118	−0.482	.8938	0.7609	−0.451	.8675	0.9090
+1.164	.7962	0.0153	−0.475	.9042	0.7643	−0.445	.8765	0.9119
+1.028	.8046	0.0180	−0.487	.9126	0.7670	−0.448	.8855	0.9148
+0.702	.8230	0.0240	−0.468	.9303	0.7728	−0.451	.9064	0.9217
+0.548	.8323	0.0270	−0.473	.9369	0.7750	−0.445	.9164	0.9249
+0.370	.8434	0.0307	−0.483	.9462	0.7780	−0.448	.9258	0.9280
+0.251	.8514	0.0333	−0.491	.9528	0.7802	−0.442	.9352	0.9311
+0.046	.8678	0.0385	−0.480	.9626	0.7834	−0.436	.9453	0.9344
−0.042	.8760	0.0413	−0.478	.9692	0.7855	−0.415	.9539	0.9372
−0.127	.8868	0.0448	−0.477	.9785	0.7885	−0.399	.9633	0.9402
−0.492	588.7692	0.3122	−0.478	.9851	0.7907	−0.363	.9720	0.9431
−0.492	.7838	0.3170	−0.477	.9949	0.7939	−0.307	.9810	0.9460
−0.488	.7918	0.3196	−0.475	820.9168	0.0948	−0.252	.9900	0.9490
−0.480	.8032	0.3233	−0.479	.9491	0.1054	−0.195	.9998	0.9522
−0.496	.8112	0.3259	−0.467	.9602	0.1090	−0.119	894.0091	0.9552
−0.492	.8220	0.3294	−0.479	.9678	0.1115	−0.491	931.7225	0.2664
−0.523	.8306	0.3322	−0.478	.9800	0.1155	−0.493	.7364	0.2709
−0.502	589.6702	0.6063	−0.483	.9935	0.1199	−0.493	.7451	0.2738
−0.443	615.6493	0.0870	−0.464	821.0033	0.1231	−0.482	.7537	0.2766
−0.455	.6650	0.0921	−0.498	837.8072	0.6086	−0.489	.7624	0.2794
−0.475	.6740	0.0951	−0.483	840.8792	0.6114	−0.487	.7711	0.2833
−0.488	617.6547	0.7417	−0.486	.8882	0.6143	−0.471	946.6451	0.1377
−0.482	.6721	0.7473	−0.481	.8952	0.6166	−0.479	.6611	0.1430
−0.464	.6797	0.7498	−0.486	.9042	0.6195	−0.467	.6704	0.1460
−0.475	.6908	0.7534	−0.488	.9105	0.6216	−0.468	.6791	0.1488
−0.491	.6988	0.7561	−0.481	.9191	0.6244	−0.438	.6881	0.1518
−0.481	.7349	0.7678	−0.479	.9257	0.6266	−0.456	.6972	0.1548
−0.491	.7450	0.7711	−0.491	.9348	0.6295	−0.491	954.7302	0.7771
−0.492	.7544	0.7742	−0.491	.9417	0.6318	−0.490	.7427	0.7812
+0.540	624.6559	0.0271	−0.487	.9702	0.6411	−0.491	.7521	0.7842
+0.330	.6712	0.0321	−0.475	.9799	0.6443	−0.492	.7618	0.7874
+0.196	.6813	0.0354	−0.487	.9869	0.6465	−0.479	.7709	0.7904
+0.069	.6920	0.0389	−0.495	.9966	0.6497	−0.483	.7802	0.7934
−0.010	.7007	0.0417	−0.498	841.0039	0.6521	−0.495	.7893	0.7964
−0.473	818.8965	0.4353	−0.493	.0139	0.6554	−0.490	.7979	0.7992
−0.495	.9104	0.4399	−0.476	860.8453	0.1292	−0.486	.8066	0.8020
−0.494	.9170	0.4420	−0.459	.8547	0.1322	−0.484	.8170	0.8054
−0.488	.9264	0.4451	−0.478	.8641	0.1353	−0.486	.8257	0.8083
−0.480	.9337	0.4475	−0.472	.8738	0.1385	−0.483	.8358	0.8116
−0.475	.9431	0.4505	−0.479	.8856	0.1423	−0.460	.8448	0.8145
−0.470	.9500	0.4528	−0.485	862.7878	0.7633	−0.503	.8538	0.8174
−0.464	.9670	0.4583	−0.487	.7964	0.7661			

$$\mathcal{J}''(y)=\frac{\mathscr{L}(y)+0.0156+0.0192\cos\theta+0.0051\cos 2\theta}{0.9891-0.0088\cos 2\theta},$$

$$Nz(y)=0.0778,$$

$$\mathcal{J}''(b)=\frac{\mathscr{L}(b)+0.0120+0.0139\cos\theta+0.0039\cos 2\theta}{0.9939-0.0017\cos 2\theta},$$

$$Nz(b)=0.0388,$$

$$\mathcal{J}''(u)=\frac{\mathscr{L}(u)+0.0153+0.0162\cos\theta+0.0050\cos 2\theta}{0.9933-0.0008\cos 2\theta},$$

$$Nz(u)=0.0444.$$

These equations are derived as if the orbit of TX UMa were a circle, but as has been proved by Russell (1912), the departures from them, if strict account be taken of the eccentric orbit, are very small.

It is well known that, except for the particular orientation with the major axis in the plane of the sky, the exact and complete solution of a light curve for stars moving in an eccentric orbit proceeds by successive approximations. The details of the method were given first by Russell and subsequently restated and somewhat elaborated by Kopal (1946). The technique involves the pairing up of the geometric and scale elements which can be derived individually from primary and sufficiently deep secondary minima so as to obtain a consistent solution. To the degree of accuracy that quantities of the second order in the orbital eccentricity can be neglected, the equations for con-

TABLE Ic. Violet observations of TX UMa.

V–C	JD (hel.) 2435000+	Phase
$-0^{\mathrm{m}}.945$	528.8466	0.7510
−0.949	.8588	0.7549
−0.935	.8674	0.7578
−0.932	.8761	0.7606
−0.945	.8851	0.7635
−0.945	530.6858	0.3514
−0.906	.6994	0.3558
−0.923	.7112	0.3597
−0.921	.7205	0.3627
−0.934	.7292	0.3655
−0.936	.7379	0.3684
−0.928	.7466	0.3712
−0.937	.7563	0.3744
−0.927	.7691	0.3786
−0.918	.7778	0.3814
−0.922	.7872	0.3845
−0.930	.7962	0.3874
−0.942	.8070	0.3909
−0.937	.8157	0.3938
−0.956	.8254	0.3969
−0.943	.8344	0.3999
−0.933	.8448	0.4033
−0.946	.8584	0.4077
−0.948	.8681	0.4109
−0.930	.8785	0.4143
−0.936	.8924	0.4188
−0.935	.9021	0.4220
−0.951	.9143	0.4260
−0.951	.9285	0.4306
−0.934	.9396	0.4342
−0.931	.9497	0.4375
−0.927	.9591	0.4406
−0.945	.9723	0.4449
+1.109	538.8380	0.0126
+0.833	.8509	0.0168
+0.581	.8627	0.0206
+0.413	.8710	0.0234
+0.147	.8853	0.0280
−0.047	.8967	0.0317
−0.216	.9082	0.0355
−0.267	.9193	0.0391
−0.608	.9425	0.0467
−0.677	.9530	0.0501
−0.805	.9745	0.0571
−0.779	541.6938	0.9448
−0.366	.7431	0.9609
−0.264	.7504	0.9633
−0.123	.7605	0.9666
+0.039	.7716	0.9702
+0.274	.7845	0.9745
+1.018	.8195	0.9859
+1.254	.8306	0.9895
+1.385	.8390	0.9922
+1.534	.8511	0.9962
+1.594	.8598	0.9990
+1.546	.8741	0.0037
+1.484	.8824	0.0064
+1.279	.8945	0.0103
+1.108	.9029	0.0131
+0.849	.9147	0.0169
+0.659	.9234	0.0198
−0.949	542.8152	0.3109
−0.947	.8251	0.3141
−0.934	543.8851	0.6602
−0.922	.8979	0.6643
−0.905	.9090	0.6680
−0.910	.9167	0.6705
−0.929	.9278	0.6741
−0.919	.9354	0.6766
−0.934	.9455	0.6799
−0.921	.9545	0.6828
−0.898	547.7891	0.9346
−0.512	.8568	0.9567
$-0^{\mathrm{m}}.394$	547.8665	0.9599
−0.285	.8742	0.9624
−0.144	.8849	0.9659
−0.054	.8926	0.9684
+0.106	.9023	0.9715
+0.267	.9099	0.9740
+0.477	.9207	0.9776
+0.638	.9283	0.9800
+0.885	.9394	0.9837
+1.019	.9481	0.9865
+1.274	.9596	0.9903
+1.453	.9693	0.9934
+1.565	.9814	0.9974
+1.592	.9915	0.0007
−0.923	551.6684	0.2009
−0.941	.6868	0.2069
−0.900	.6966	0.2101
−0.906	.7091	0.2142
−0.930	.7177	0.2170
−0.938	.7288	0.2207
−0.937	.7361	0.2230
−0.928	.7469	0.2266
−0.935	.7750	0.2357
−0.923	.7816	0.2379
−0.930	.7907	0.2409
−0.931	.8393	0.2567
−0.942	.8486	0.2598
−0.945	.8559	0.2622
−0.894	552.6194	0.5114
−0.911	.6354	0.5166
−0.903	.6424	0.5189
−0.889	.6517	0.5219
−0.925	.6604	0.5248
−0.914	.6705	0.5281
−0.914	.6778	0.5305
−0.917	.6868	0.5334
−0.929	.6934	0.5355
−0.928	.7028	0.5386
−0.925	.7097	0.5409
−0.922	.7163	0.5430
−0.931	.7264	0.5463
−0.936	.7392	0.5505
−0.936	.7503	0.5541
−0.935	.7573	0.5564
−0.931	.7674	0.5597
−0.928	.7746	0.5621
−0.912	.7844	0.5653
−0.940	.7917	0.5676
−0.929	.8014	0.5708
−0.930	.8083	0.5731
−0.910	554.6207	0.1647
−0.901	.6336	0.1689
−0.895	.6402	0.1711
−0.924	.6495	0.1741
−0.922	.6561	0.1763
−0.936	.6662	0.1796
−0.926	.6728	0.1817
−0.925	.6846	0.1856
−0.922	.6909	0.1876
−0.918	.6999	0.1906
−0.911	.7065	0.1927
−0.913	.7166	0.1960
−0.937	.7298	0.2003
−0.903	.7395	0.2033
−0.905	.7502	0.2070
−0.924	.7572	0.2093
−0.923	.7669	0.2124
−0.930	.7738	0.2147
−0.917	.7853	0.2184
−0.925	.7929	0.2209
−0.909	555.6182	0.4903
−0.906	.6318	0.4948
−0.914	.6391	0.4972
$-0^{\mathrm{m}}.885$	555.6495	0.5006
−0.899	.6564	0.5028
−0.903	.6658	0.5059
−0.890	.6728	0.5082
−0.914	.6821	0.5112
−0.897	.6901	0.5138
−0.916	.7005	0.5172
−0.902	.7078	0.5196
−0.892	.7200	0.5236
−0.905	.7269	0.5258
−0.911	.7377	0.5293
−0.931	.7526	0.5342
−0.941	.7603	0.5367
−0.925	.7700	0.5399
−0.933	.7797	0.5431
−0.947	.7894	0.5462
−0.934	.7984	0.5492
−0.945	.8103	0.5530
−0.938	.8175	0.5554
−0.934	.8280	0.5588
−0.931	.8353	0.5612
−0.937	.8450	0.5644
−0.942	.8526	0.5669
−0.943	.8627	0.5702
−0.945	.8700	0.5725
−0.938	.8797	0.5757
−0.940	.8870	0.5781
−0.941	.8981	0.5817
−0.905	556.7171	0.8491
−0.936	.7317	0.8538
−0.919	.7390	0.8562
−0.921	.7484	0.8593
−0.929	.7557	0.8617
−0.913	.7671	0.8654
−0.908	.7755	0.8681
−0.920	.7862	0.8716
−0.910	.7953	0.8746
−0.923	.8046	0.8776
−0.914	.8102	0.8795
−0.912	.8248	0.8842
−0.924	.8335	0.8871
−0.900	557.7726	0.1936
−0.909	.7865	0.1982
−0.907	.7935	0.2006
−0.931	.8022	0.2033
−0.906	559.7548	0.8407
−0.907	.7691	0.8454
−0.908	.7760	0.8476
−0.928	.7861	0.8509
−0.922	.7941	0.8536
−0.916	.8034	0.8566
−0.912	.8100	0.8587
−0.876	569.6212	0.0614
−0.904	.6372	0.0667
−0.900	.6442	0.0690
−0.890	.6532	0.0719
−0.894	.6619	0.0748
−0.886	.6712	0.0778
−0.880	.6778	0.0800
−0.903	.7070	0.0895
−0.912	.7178	0.0930
−0.912	.7247	0.0953
−0.898	.7341	0.0983
−0.895	.7424	0.1011
−0.906	.7528	0.1044
−0.900	.7883	0.1160
−0.912	.7973	0.1190
−0.941	579.6449	0.3337
−0.940	.6595	0.3384
−0.946	.6872	0.3475
−0.958	.6976	0.3509
−0.111	584.6471	0.9666
+0.149	.6641	0.9721

Table Ic.—*Continued.*

V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase	V–C	JD (hel.) 2435000+	Phase
+0^{m}455	584.6803	0.9774	−0^{m}938	818.9507	0.4530	−0^{m}941	862.7971	0.7663
+0.598	.6879	0.9799	−0.936	.9677	0.4586	−0.947	.8058	0.7691
+0.842	.6983	0.9833	−0.941	.9744	0.4608	−0.927	.8145	0.7720
+1.091	.7087	0.9867	−0.950	.9844	0.4640	−0.935	.8232	0.7748
+1.279	.7191	0.9901	−0.940	.9917	0.4664	−0.921	893.8275	0.8959
+1.443	.7282	0.9931	−0.947	819.0024	0.4699	−0.898	.8418	0.9006
+1.530	.7382	0.9963	−0.945	.0118	0.4729	−0.904	.8505	0.9034
+1.559	.7455	0.9987	−0.939	.0292	0.4786	−0.907	.8591	0.9062
+1.609	.7577	0.0027	−0.936	.0375	0.4814	−0.910	.8682	0.9092
+1.453	.7684	0.0062	−0.925	.8945	0.7611	−0.912	.8772	0.9121
+1.307	.7792	0.0097	−0.928	.9049	0.7645	−0.905	.8862	0.9151
+1.156	.7862	0.0120	−0.928	.9133	0.7673	−0.906	.9071	0.9219
+0.914	.7969	0.0155	−0.917	.9310	0.7730	−0.902	.9171	0.9252
+0.741	.8053	0.0182	−0.927	.9376	0.7752	−0.898	.9265	0.9282
+0.354	.8237	0.0242	−0.925	.9469	0.7782	−0.898	.9359	0.9313
+0.162	.8330	0.0273	−0.933	.9535	0.7804	−0.885	.9459	0.9346
−0.013	.8441	0.0309	−0.937	.9633	0.7836	−0.862	.9546	0.9374
−0.148	.8521	0.0335	−0.933	.9699	0.7857	−0.827	.9640	0.9405
−0.371	.8695	0.0392	−0.930	.9792	0.7888	−0.788	.9727	0.9433
−0.471	.8778	0.0419	−0.931	.9858	0.7909	−0.729	.9817	0.9462
−0.555	.8886	0.0454	−0.935	.9956	0.7941	−0.655	.9907	0.9492
−0.945	588.7699	0.3124	−0.898	820.9175	0.0951	−0.588	894.0005	0.9524
−0.949	.7845	0.3172	−0.927	.9498	0.1056	−0.516	.0098	0.9554
−0.947	.7925	0.3198	−0.893	.9609	0.1092	−0.935	931.7232	0.2666
−0.928	.8039	0.3235	−0.917	.9685	0.1117	−0.941	.7371	0.2712
−0.957	.8119	0.3261	−0.915	.9807	0.1157	−0.946	.7457	0.2740
−0.959	.8237	0.3300	−0.926	.9942	0.1201	−0.934	.7544	0.2768
−0.975	.8320	0.3327	−0.923	821.0040	0.1236	−0.940	.7631	0.2796
−0.960	589.6709	0.6065	−0.938	837.8079	0.6088	−0.943	.7718	0.2825
−0.903	615.6500	0.0872	−0.929	840.8799	0.6116	−0.922	946.6458	0.1380
−0.943	.6656	0.0923	−0.934	.8889	0.6145	−0.914	.6618	0.1432
−0.926	.6761	0.0958	−0.944	.8959	0.6168	−0.918	.6711	0.1462
−0.950	617.6561	0.7421	−0.950	.9049	0.6198	−0.916	.6798	0.1491
−0.938	.6728	0.7476	−0.944	.9112	0.6218	−0.910	.6888	0.1520
−0.926	.6804	0.7501	−0.936	.9198	0.6246	−0.922	.6979	0.1550
−0.951	.6901	0.7532	−0.937	.9264	0.6268	−0.931	954.7309	0.7773
−0.940	.6995	0.7563	−0.944	.9355	0.6298	−0.924	.7434	0.7814
−0.933	.7356	0.7681	−0.942	.9424	0.6320	−0.937	.7528	0.7845
−0.931	.7457	0.7714	−0.942	.9709	0.6413	−0.925	.7625	0.7876
−0.941	.7551	0.7744	−0.928	.9806	0.6445	−0.928	.7716	0.7906
+0.222	624.6542	0.0266	−0.946	.9875	0.6467	−0.920	.7809	0.7936
−0.079	.6726	0.0326	−0.940	.9973	0.6499	−0.933	.7900	0.7966
−0.227	.6830	0.0360	−0.958	841.0046	0.6523	−0.937	.7986	0.7994
−0.352	.6938	0.0395	−0.953	.0146	0.6556	−0.920	.8073	0.8023
−0.941	818.8972	0.4356	−0.926	860.8460	0.1294	−0.929	.8177	0.8056
−0.957	.9111	0.4401	−0.909	.8554	0.1326	−0.922	.8264	0.8085
−0.951	.9177	0.4422	−0.929	.8648	0.1356	−0.920	.8365	0.8118
−0.942	.9271	0.4453	−0.912	.8745	0.1387	−0.913	.8455	0.8147
−0.944	.9344	0.4477	−0.920	.8863	0.1424	−0.909	.8545	0.8177
−0.944	.9437	0.4507	−0.921	862.7884	0.7636			

verting the elements derived from the solution of one eclipse to those which should follow from the solution of the other eclipse involve the eccentricity itself and the longitude of periastron. It thus becomes important to know the aspect of the orbit at the time the observations were taken. From one point of view this follows most conveniently from the displacement of secondary eclipse from the half-period point. The previous investigations of TX UMa by O'Keefe and Wood indicate the locations of secondary eclipse to be about at the expected phases. Inspection of the shallow minimum in the Tucson light curves, illustrated in Fig. 2, shows that the

Table II. Magnitudes and color indices of comparison stars and TX UMa.

Star	*V*	*B−V*	*U−B*	*n*
+47°1797	+7^{m}38±0^{m}02	+0^{m}15±0^{m}01	+0^{m}08±0^{m}01	10
+47°1799	8.56±0.02	+0.12±0.01	+0.09±0.01	10
TX UMa, 0^{P}00	8.76	+0.47	+0.01	. . .
TX UMa, 0.25	7.06	+0.01	−0.36	. . .
TX UMa, 0.50	7.13	−0.02	−0.37	. . .
TX UMa, 0.75	+7.06	+0.01	−0.36	. . .

Table III. Provisional yellow solution of TX UMa.

$x=0.6$ $j=82\overset{\circ}{.}3$
$k=0.615$ $\alpha_0^{oc}=0.945$
$a_g=0.270$ $L_g=0.196$
$a_s=0.166$ $L_s=0.804$

lipse is displaced in the proper direction and that its cation seems not to be a function of the wavelength observation. Detailed study of the observations fails, owever, to convince one that the lowest point of lipse occurs at the predicted phase. Indeed, it seems s if the eclipse is displaced only about half the expected eparture and that the light curve varies conspicuously shape within eclipse. As a consequence, information erived from the analysis of secondary minimum must e viewed with considerable reservation. More than kely, the eccentricity of the orbit and the longitude of eriastron are really the values which are expected from he discussions of times of primary minimum, and omehow secondary eclipse is distorted in shape and erhaps in depth. It should not be surprising, therefore, inconsistency arises between the separate analyses or the elements from the two eclipses. For the present rientation of the line of apsides the effect of orbital ccentricity upon the eclipses creates only a small eparture from a circular solution.

Each of the yellow, blue, and violet light curves was ndividually rectified for darkenings of 0.2, 0.4, 0.6, nd 0.8. Trials using the Princeton nomographs revealed he solution for each light curve to be similar to that ound by previous investigators. Attempts to refine the omograph solutions were also made in each instance, nd for this purpose the ψ tables of Merrill (1950) were sed. By and large, these attempts were successful, and elected elements of the provisional yellow solution, eferred to the eccentric orbit, are listed in Table III. t should be remarked that the determination of the imb darkening coefficient is not strong. The fit of the olution suggests that a relatively large darkening gives he best results, and this information is substantiated by he coefficients predicted by Münch and Chandrasekhar 1949). A more detailed comparison of theory and obervation by Hosokawa (1957a) suggests further that a nean coefficient of 0.6 works just as well as the individual ones which follow from his theory.

Solutions for the blue and violet observations were lso obtained using the ψ tables. In general, these etermine geometric and photometric elements which re similar to those determined from the yellow observations. There are, however, disturbing discordances which prevent exact consistency of one light solution with another. As a consequence of this, it was decided o test the mean solution against the theoretical Russell model and thus seek some clue to the sources of the discordances.

From the depth of primary eclipse used in the preliminary solutions at each wavelength and from the *U,B,V* photometry of the comparison star it is possible to derive the color indices of the components of TX UMa. This is shown in Table IV, in which the first column lists the stars individually; the second and third columns list the color indices of the interior, facing hemispheres; and the fourth and fifth columns give the color indices for the exterior, remote hemispheres. It can be seen that, in general, the interior hemispheres are bluer (i.e., hotter) than the outside ones. The differences are greater in $B-V$ than in $U-B$, and they are greater for the cool star than for the hot star. The hot star seems clearly to exhibit colors associated with B stars, suggesting that the spectral classification of B8 is well established. The colors given for the cool component place the range of possible spectral class from F8 to G2. In all probability, the luminosity class of this component is that of a subgiant, as can be seen from the ratio of the radii determined from the provisional solution. These conclusions are in general agreement with those of Wood, who found the ratio of the eclipse depths in blue light to determine a spectral type of about F9 for the cool star. He suggests that the difference between the spectroscopic classification of gF2 and that of a late F star is significant.

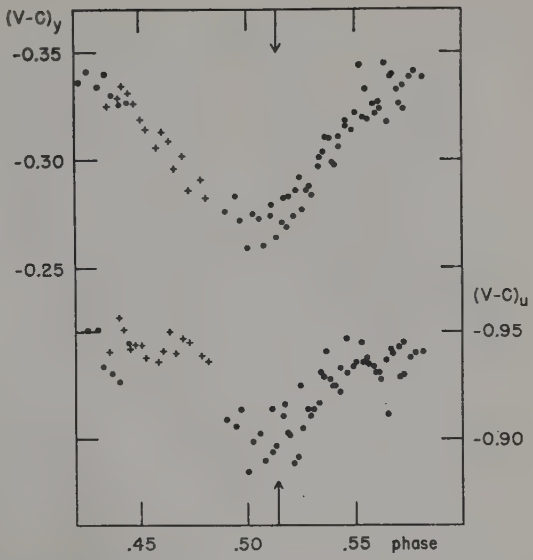

Fig. 2. Yellow and violet observations of TX UMa within secondary eclipse. The filled circles refer to observations of the first season and the plus signs to observations of the second season. The phases of the observations have been computed from the ephemeris of Sec. III. The theoretical location of secondary minimum computed from the same ephemeris is indicated by the arrows.

Table IV. Preliminary color indices of components of TX UMa.

Star	$B-V$	$U-B$	$(B-V)'$	$(U-B)'$
Hot	$-0^{\mathrm{m}}10$	$-0^{\mathrm{m}}40$	$-0^{\mathrm{m}}08$	$-0^{\mathrm{m}}39$
Cool	$+0.62$	$+0.04$	$+0.70$	$+0.04$

Table V. Reflection effect in $\cos\theta$ for TX UMa.

Light curve	$(E_h/E_c)_R'$	$(E_h/E_c)_H'$	A_{1R}'	A_{1H}'	A_1	$(E_h/E_c)''$	$(E_h/E_c)'''$
Yellow	0.62	0.54	−0.037	−0.042	−0.019	1.08	1.06
Blue	1.15	1.10	0.021	0.022	0.014	1.59	1.54
Violet	1.74	1.35	−0.014	−0.019	−0.016	1.49	1.46

With the spectral types of the stars known more surely than at the beginning of the solution, it is possible to return to the outside-eclipse variations and compute the theoretical coefficient of $\cos\theta$ at each wavelength. This coefficient is defined as

$$A_1 = -0.40[L_h r_c^2(E_c/E_h) - L_c r_h^2(E_h/E_c)] \sin i.$$

In this equation r refers to the mean radius of a component; L refers to the luminosity of its interior hemisphere; and E to the luminous efficiency of its surface as it reradiates the light that falls upon it from the companion star. The L's and r's follow from the provisional solutions, the yellow values being given in Table III, and the luminous efficiencies are given by theory. At present, two theories for the computation of E are available. There is, first of all, that developed by Russell and Pismis and quoted in Russell and Merrill (1952c) which depends upon graybody radiation for the star. There is, in addition, the theory developed by Hosokawa (1957b), which takes account of the departures of the stellar radiation from graybody character. The computed ratios of the luminous efficiencies are shown in Table V, where the subscripts H and R refer to the values determined by Hosokawa's and Russell's theories, respectively. From these computed ratios the theoretical values of A_1 have been calculated. They are shown in columns four and five of Table V and are contrasted with the observed values of A_1 rounded off to three decimals. It can be seen that the theoretical values of the differential reflection coefficient differ significantly from the observed values and that the discordance becomes worse as one goes to longer wavelengths. This is a situation which is, at least in part, familiar to many observers of eclipsing stars. For instance, a similar situation has been reported for Algol by Hosokawa (1960). It is also well known that in particularly close binaries, such as those of the W UMa type, the coefficient of reflection is frequently positive in contradiction to the theoretically negative value. The situation in TX Uma, at least for yellow and blue light, is similar in that the observed values are more positive than the theoretical ones. The main assumptions which underlie the theory of reflection as it is presently applied are that the stars are spheres and that each is in perfect radiative equilibrium. There are, however, additional secondary effects which can be treated only in approximation. Among these are the quantitative allowances for the finite separation and sizes of the components together with penumbral effects of illumination. No improvement in the treatment of these being expected at present, it seems that the observed departures should be attributed to a grossly incorrect solution or to violations of at least one of the two main assumptions.

Fortunately, the terms which make up the expression for A_1 are at least partly separable so that they can be checked by independent means. If we suppose first of all that it is the relative luminosities of the stars which are in error, it is possible to juggle these so as to reproduce the observed reflection coefficients. This occurs in blue light when L_h is approximately 0.7 and L_c is 0.3. This explanation, however, cannot be the correct one, for outside eclipse we see the absorption lines of only the hot star. It is possible that the radii of the stars given by the preliminary solution are in error. These, however, agree satisfactorily with previous determinations.

It is difficult to see how a departure from spherical figure can decrease the amount of light which is reflected toward the observer. As the stars are distorted by their rotations and mutual gravitational attraction, the interior hemisphere of each is elongated more than the exterior side. This interior hemisphere, being the main seat of reflected radiation, will therefore be larger than in an undistorted sphere, and the total amount of reflected light, if it obeys the distribution law given by Milne (1926) and others, should be greater than that reflected from a spherical star. More than likely, there are appreciable departures from radiative equilibrium and theoretical luminous efficiency within the atmospheres of the stars. In particular, the cool star of larger size and lower surface gravity is suspect. A different conclusion has been expressed by Hosokawa (1957b), who suggested that the source of this same difficulty, encountered in his statistical study, can be traced to incorrect values of E for B stars. This conclusion, however, is not consistent with the situation in TX UMa, for the discordance in A_1 becomes greater as the luminosity of the cool star contributes a larger fraction of the light of the binary. It seems more likely, therefore, that the efficiencies which are incorrect are those associated with the cool component. This being so, it is possible to determine the ratio, E_h/E_c, from the observed luminosities and radii of the stars given by the preliminary solutions and the observed values of the differential reflection coefficient. These forced ratios of luminous efficiency are given in the last column of Table V. If we take at face value the theoretical efficiences tabulated for a B8 star, then those determined for the cool component by the observed reflection coefficients are ones associated with a G or K star. To sum p, the view presented asserts that a preliminary olution based upon either graybody or nongraybody heory will give an informative comparison of the heoretical and observed reflection coefficients. Should hese be found to differ, the ratio of the luminous fficiencies as a function of wavelength may be determined from the observed reflection effect. It follows hat the luminous efficiency is probably a function of he sizes and the separation of the stars and should be etermined individually for each eclipsing binary. This ppears to be not too difficult to accept, for the theoetical values computed according to either one of the heories pertain to single, isolated stars.

It is now necessary to compare the observed and heoretically expected values of the coefficient of cos2θ, onventionally A_2, in the observed light curves. This oefficient is predominantly determined by the ellipicities of figure of the stars, but there is, in addition, a mall term contributed by the reflection effect. The omplete expression is given, for instance, by Hosokawa (1957a) in his Eq. (17). For the purposes of the present nvestigation, his final term standing for the small ourth-order distortion of figure is neglected. The mean hotometric ellipticities follow from the observed utside-eclipse variations, and these have been used for the preliminary solutions of the eclipses themselves. As has been pointed out by Dr. J. E. Merrill in private conversation, it is almost certain that the two components of TX UMa differ significantly in their figures. This assertion can be examined by the tests given by Russell and Merrill (1952d). At the University of Pennsylvania, Merrill and Reuning have calculated, but not yet published, the radii in four directions for the $n=3$ polytropes of Chandrasekhar (1933). From the mass ratio given by Pearce (1933) and Wood and the radii given by Merrill and Reuning it is possible by interpolation and iteration to recover the radius of the undistorted polytrope and then compute the two equatorial radii in the direction of the line of centers, the equatorial radius in the plane perpendicular to the line of centers, and the polar radius. It is found that for each component of TX UMa the radii in the direction of the line of centers are very nearly equal. Finally, it is possible to compute the equatorial and polar ellipticities of figure, customarily represented as z. For the hot star the two ellipticities are each equal to 0.011. For the cool star the equatorial ellipticity is 0.187, while the polar ellipticity is 0.093. It is thus confirmed that the two stars are distinctly different in figure, and that the cool star is distorted from an ellipsoid of rotation. At this stage of the computation it seemed permissable to work with a mean value of the ellipticity of the cool star, and this was taken as 0.139.

In order to compute the expected value of the coefficient of cos2θ, it is necessary to make some assumption about the ratio of photometric ellipticity to geometric ellipticity. This is a quantity, little known today, which depends upon the gravity and limb darkenings of a star. Investigation by Hosokawa (1957a) again shows that, in general, one can do little better than choose a value for this ratio appropriate to a limb darkening of 0.6. From these quantities and the ratios of the luminous efficiencies, the values of A_2 have been computed and are listed in Table VI as a function of wavelength. For this computation the disparities between the luminous efficiencies of Russell and Hosokawa are of no significance because the ratios are multiplied by a small number. These theoretical values are given as A_2' in Table VI, while the values computed with the forced luminous efficiencies are given as A_2''. The observed values are listed as A_2. It can be seen that the change in the luminous efficiencies for the violet curve produced no change in the computed ellipticity effect. For the blue curve the agreement with the new luminous efficiencies is slightly worse than it was previously, but for the yellow curve the agreement is now strikingly better. This is taken as evidence that forcing the ratios of luminous efficiency in order to fit the individual binary is a change in the proper direction.

TABLE VI. Reflection and ellipticity effect in cos2θ for TX UMa.

Light curve	A_2'	A_2''	A_2	A_2'''
Yellow	−0.007	−0.014	−0.014	−0.014
Blue	0.007	0.010	0.006	0.014
Violet	−0.006	−0.006	−0.006	−0.010

The elements of the system being provisionally known, it is possible to compute the theoretically expected coefficients of the higher-order terms in the reflection and ellipticity effects. These depend upon the luminosities of the individual stars and their radii and the mass ratio. The coefficient of the third-order term for TX UMa is computed to be about −0.001 though there is a slight variation with wavelength as a consequence of the dependence of the luminosities on wavelength. The fourth harmonic of the ellipticity and reflection effects is computed to be 0.000.

The attitude having been taken that the ratios of the luminous efficiencies of the components of TX UMa have been improved, it was decided to re-rectify the individual observations, and accordingly the outside-eclipse variations were inspected once again. It was concluded that any improvements to be made in the coefficients were small and would not affect the solutions significantly. At this time it was found useful to change by a small amount the normalizing magnitude differences to be added to the blue and violet observations and thus change these outside-eclipse variations by a factor close to unity. These new variations are given as

$$\mathfrak{L}(b)=0.9852-0.0139\cos\theta-0.0056\cos2\theta,$$

$$\mathfrak{L}(u)=0.9846-0.0163\cos\theta-0.0059\cos2\theta.$$

The new coefficients for rectification are given in

Table VII. Improved rectification coefficients for TX UMa.

Light curve	C_0	C_1	C_2	Nz
Yellow	+0.0185	+0.0192	+0.0059	+0.0812
Blue	0.0135	0.0139	0.0043	0.0407
Violet	+0.0143	+0.0163	+0.0047	+0.0432

Table VII. As anticipated, changes in the rectified observations are small.

A determinate, consistent solution from the combination of both eclipses was now sought. It was found that, while at each wavelength there is indeed a determinate solution defined by the depth of both eclipses and the shape of primary, the solutions so discovered are not identical and cannot be made identical within the observational error by changing the darkening coefficient. At least some of these difficulties can be traced to the shape of primary eclipse, which varies with wavelength. For a second trial it was decided to seek the best common solution defined only by the depth lines for all wavelengths. Again, it was found that no consistent, recognizable, partial solution results so that the observed depth of at least one of the eclipses is sensibly incorrect in at least one wavelength. Since the depths of primary eclipse give $B-V$ and $U-B$ color indices that are reasonably appropriate for a B8 star, it seems likely that a large percentage of the discrepancies is to be traced to incorrect depths of secondary. In order to test this, the color indices for the cool component were derived from the depths of secondary eclipse. These were found to be $0\overset{\mathrm{m}}{.}2$ to $0\overset{\mathrm{m}}{.}3$ bluer than should be the case for a late F-type star. As a result, it seems certain that the observed depth of secondary eclipse is significantly incorrect in at least two wavelengths and one would like to know which of the observed depths is most consistent with primary eclipse. This it is also possible to find, and three trials determined that the depth of secondary eclipse observed in violet light is most nearly correct. The departures of the secondary eclipse depths from what they should be seem to be correlated with the departures of the reflection coefficients from the theoretical values. This occurs in the sense that the observed reflection coefficients make the depths of secondary too shallow in yellow and blue light.

So many difficulties having been detailed already, it may be useful to summarize them at this point. First, though the depths of primary eclipse appear reasonably accordant, it seems certain that there are small disparities in the shapes which make it virtually impossible to determine a unique, consistent solution. Second, between eclipses the light variations differ from those computed on the basis of the Russell model. Agreement may be somewhat improved if empirically determined luminous efficiencies are used instead of those which follow from blackbody, graybody, and nongraybody theories. Third, the depth of secondary eclipse i certainly not what theory predicts in yellow and blu light, but agreement is satisfactory in violet light Finally, the shapes of the branches of secondary eclips suggest strongly that light variations may be prominen at this phase in the light curve and that they too may be a function of wavelength. These discrepancies being understood to inhibit the determination of any solution it seems the most logical procedure to rely on the depths and shapes of primary eclipse as far as they can be brought to consistency and on the observed depth of the violet secondary eclipse.

VI. FINAL SOLUTION OF PRIMARY ECLIPSE

It has been noted by Hiltner (1945) that the K line of calcium in the spectrum of TX UMa seems to show characteristics traceable to an atmospheric eclipse. His conclusion, while it cannot be validated for the time of this photometry, suggests that there may be present in the 1956–1957 observations additional atmospheric absorption capable of distorting the shapes and the depths of at least one of the eclipses. It was finally decided to treat the yellow observations in primary for a body eclipse involving sharp-edged stars but to permit a sharp-edged, extensive atmosphere to distort this model for the blue and violet observations. The atmosphere is conceived to absorb selectively, but it may also absorb and scatter in a nonselective fashion. Though this cannot be proved to be the correct attack, it is certainly true that the absorption lines normally present in stellar spectra are more numerous toward shorter wavelengths, so that atmospheric effects should be enhanced in passing from yellow to blue to violet observations. To make such a solution it is necessary to depart from the Russell model in favor of one similar to that expounded by Kron and Gordon (1950) in their study of V444 Cyg. In their model both stars are sharp-edged, but one is surrounded by at least a less dense shell. In projection the rectified model exhibits two circular stellar disks which may be darkened to the limb in the ordinary manner, and the component possessing the extensive atmosphere is surrounded by an annulus which is concentric with the star and touches it at its photosphere. The equations which permit the solution of primary eclipse, in the case of TX UMa the partially atmospheric occultation eclipse, are given below for equal, unspecified darkenings in yellow and blue. In these equations the unprimed symbols refer to characteristics of the photospheres of the components. The singly primed symbols refer to the disk described by Kron and Gordon, and the doubly primed symbols refer to the projected atmospheric annulus. The equations for yellow and violet are similar, and the extension to dissimilar darkenings is performed by the usual technique. The arrows indicate the sequence of calculating the succeeding quantities.

$$(\alpha,k,p)_y{}^{oc}=(\alpha,k,p)_b{}^{oc}\rightarrow\delta,$$
$$(\delta=\delta',\,k')_b\rightarrow p_b'\rightarrow\alpha_b{}'^{\,oc},\quad k'=k'',$$
$$\alpha_b{}''^{\,oc}=\alpha_b{}'^{\,oc}-\alpha_b{}^{oc},$$
$$(1-\ell)_b{}^{oc}+(1-\ell'')_b{}^{oc}=d,$$
$$(1-\ell_0)_b{}^{oc}=(\alpha_{0,b}{}^{oc})(L_{h,b}),$$
$$(1-\ell_0'')_b{}^{oc}=(1-f_b)(\alpha_{0,b}{}''^{\,oc})(L_{h,b})\rightarrow f_b.$$

Here d equals the observed depth of primary eclipse in blue light at any phase.

The solution is begun by assuming provisional amounts of atmospheric absorption in blue and violet and is continued by successive trial values for the longitude of external tangency, $(1-\ell_0'')^{oc}$, and a_g''. It was decided that physical reality is best served by assuming the limb darkening coefficient in blue and violet to be not less than that chosen in yellow. The best yellow solution is tabulated in Table VIII. For 0.6 limb darkening it proved impossible to fit the blue and violet observations with a consistent body and atmospheric eclipse as well as the observations suggested might be done. Trial solutions for both blue and violet light were next computed assuming 0.6 limb darkening for the photospheres and 0.8 darkening for the atmospheres. These proved as inferior as the previous 0.6 solutions. Solutions were obtained, however, when the blue and violet observations were treated for 0.8 darkening both for the disks and for the atmospheres; refined through several trials, these also appear in Table VIII. In those characteristics which refer to the sharp-edged photospheres the solutions at all wavelengths are now consistent. The light curves were analyzed on a scale 1 cm to 0.01 in rectified light. Reduced to the size of a printed figure the differences between unsatisfactory and satisfactory solutions cannot be distinguished. For a single observation the probable residual from the final solution is $\pm 0^{\mathrm{m}}.010$ in yellow, $\pm 0^{\mathrm{m}}.009$ in blue, and $\pm 0^{\mathrm{m}}.008$ in violet light.

Table VIII. Final solution for primary eclipse of TX UMa.

$k=0.615$
$a_g=0.270$
$b_g=0.267$
$a_s=0.166$
$b_s=0.164$
$j=82^{\circ}.2$

$V_h'=+7^{\mathrm{m}}.32$
$V_c'=+9.00$
$(B-V)_h'=-0.09$
$(B-V)_c'=+0.58$
$(U-B)_h'=-0.41$
$(U-B)_c'=+0.21$

	y	b	u
$(1-\ell_0)^{oc}$	0.7570	0.8296	0.8647
x	0.6	0.8	0.8
$\alpha_0{}^{oc}$	0.945	0.953	0.953
$L_h=L_s$	0.801	0.871	0.908
$L_c=L_g$	0.199	0.129	0.092
z	0.031	0.013	0.014
L_h'	0.796	0.866	0.905
L_c'	0.156	0.098	0.057
$(J_h'/J_c')_{av}$	13.5	23.4	41.8
$(1-\ell_0'')^{pr}$	...	0.0014	0.0033
x''	...	0.8	0.8
$\alpha_0{}''^{\,oc}$	...	0.031	0.031
f	...	0.948	0.883
a_g''	...	0.285	0.285

The large residual in yellow is caused, in large part, by the scatter of the observations at the bottom of eclipse.

The final values of $(E_h/E_c)'''$ and A_2''' are given in Tables V and VI, respectively. The former have changed little. The latter are now too negative in blue and violet, a situation that can be traced to the necessity for 0.8 limb darkening at these wavelengths.

VII. DISCUSSION

It is highly desirable to comment upon the determinancy of the elements listed in Table VIII. While extensive treatment of the yellow observations demonstrated that darkenings of 0.2 and 0.8 were not the best that could be desired, it proved impossible to discriminate between values of 0.4 and 0.6. The solution thus rests on the assumption that the true coefficient is 0.6. The difference of darkening between the yellow and blue or violet sets of observations is undeniable, but it has not been proved that this difference would exist if both components were normal, well-separated stars. In other words, it may be that the observed difference is an artifact of the complicating, atmospheric effects that we see in primary eclipse. As an example of the determinancy of the yellow solution of primary itself, it can be remarked that the following solutions on the assumption of a 0.6 limb darkening were found to be distinctly inferior: $k=0.602$, $\alpha_0{}^{oc}=0.955$; $k=0.620$, $\alpha_1{}^{oc}=0.900$; and $k=0.635$, $\alpha_0{}^{oc}=0.895$. These were not the only solutions so tested, but they are the ones closest to the final solution. There is a certain amount of latitude in determining the best atmospheric solutions in the blue and violet. Trial solutions serve to indicate, however, that a_g'', for instance, is not 0.277 or 0.293. It is legitimate to conclude then that this particular element is known to an accuracy of about 3%. There is no physical constraint which necessitates that a_g'' be the same in blue and violet light, and probably a difference could be detected with more accurate observations. The eclipse observations of the second season, while few in number and covering but the top portion of one of the shoulders of the light curve, demonstrate that primary minimum varies in shape. Furthermore, the departures of the observations of this season from those of the first one are greater in blue and violet than they are in yellow. One is tempted to suspect, therefore, continuing activity associated with the cool component which can be treated to a reasonable degree of accuracy as growth and decay of an extensive envelope. It is quite likely that there is a certain amount of atmospheric absorption present in the yellow light curve. In the absence of extensive spectroscopic observations in the visual region this feature remains unknown.

From the limiting solutions given by Wood the radius of the cool star seems reasonably well known, so that the depth of the atmosphere can be calculated. This is 1.53×10^5 km. The line of sight from the observer to the disk of the hot star traverses at most 1.86×10^6 km through the extensive atmosphere of the cool star.

It was thought at first that these data, together with plausible values of the mean molecular weight and temperature in the atmosphere, could provide some information about the density gradient with atmospheric height. Because of the small number of the observations and the inevitable inaccuracies of these, this expectation has not been borne out.

It should finally be remarked that the observations were tested against the atmospheric tables of Linnell (*Astrophys. J.* to be published). These tables permit the application of five different density gradients in an extensive atmosphere. It was found that none of these could be made to represent the observations of TX UMa, thus indicating the presence of a photosphere for the cool component.

VIII. SECONDARY ECLIPSE

Since the use of the observed coefficients of reflection gave disparate colors for the cool star, it seemed useful to rectify at least secondary eclipse according to the average theoretical values of A_1 given in Table V. This treatment of these observations by the admission of a complication term in $\cos\theta$ is not completely successful for two reasons. First, the coefficient of $\cos 2\theta$ to be used in the rectification for ellipticity in blue light comes out to be $+0.0005$ and thus of the wrong algebraic sign. Second, the shape of each branch of the minimum is still inconsistent with the solution from primary eclipse. There is, however, a strong indication that this piecemeal rectification is the correct procedure to follow, for the colors of the cool star as they can now be derived from the rectified depths of secondary are those of a late F-type star as were the colors derived from primary eclipse.

Though this *ad hoc* method of treating secondary eclipse can do nothing about the strongly asymmetric minimum at each wavelength, it does suggest a reasonably coherent model for TX UMa. A partially absorbing, re-emitting, and fluorescing atmosphere surrounds the cool star. Its presence and the activity within it create departures from the theoretical reflection effect and the ellipticity effect insofar as the latter depends on the observed reflection and limb darkening. At secondary eclipse the uneclipsed fraction of the atmosphere is greater than the eclipsed portion, causing $(1-\ell)^{tr}$ to be too small. The variable extent and density of the atmosphere cause changes in the depths and shapes of the eclipses, but the changes in secondary eclipse are proportionately more pronounced.

IX. SUMMARY

It should be fairly clear that the solution given in Table VIII cannot be considered the definitive one for TX UMa, but it does emphasize a few points whic bear on the treatment of all eclipsing variables. Particularly for those in which one component nearly fill the inner Lagrangian surface, observation at only on wavelength offers no assurance that correct element can be determined. This forces us to view with considerable reservation orbital solutions which rest o even well-covered visual or photographic light curves. At best, these provide an instantaneous photometric solution.

Finally, if the combination of body and atmospheric eclipses is shown to have extensive application, it might be profitable to formalize the Kron model more elegantly and compute tables for it.

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The Constitutions of Uranus and Neptune

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The internal constitutions of Uranus and Neptune have been investigated on the basis of the assumption that solid ammonium is a major constituent of these planets. An equation of state found by Bernal and Massey for this substance has been employed together with the equation of hydrostatic equilibrium to find the internal density and mass distributions. The presence of undetermined, more dense materials has been allowed for in a phenomenological manner and it has been found that the ammonium contents are 84 and 74% for Uranus and Neptune, respectively. Both planets appear to have dense cores of heavy material of approximately twice the mass of the earth.

I. INTRODUCTION

It is the purpose of this paper to report the results of an investigation of the internal structure of Uranus and Neptune based upon the hypothesis that metallic ammonium is a major constituent of these planets. This possibility has been pointed out by Ramsey (1951) who, in a study of the major planets, showed that Uranus and Neptune cannot be composed primarily of solid hydrogen, as are Jupiter and Saturn, but are likely to contain large amounts of water, ammonia, or methane. Bernal and Massey (1954) have shown that ammonia (NH_3) and hydrogen will combine to form stable ammonium metal (NH_4) at pressures less than those which prevail throughout most of the interiors of Uranus and Neptune, and have thus given further support to Ramsey's suggestion. The former workers have also calculated by quantum-mechanical methods the 0°K isotherm of the metal and a quantitative study of these planets may be made.

The only current models of the two smaller major planets are those of Wildt (1947), who has assumed that they were composed of three layers of material, each layer or shell having a uniform density. When reasonable values were assigned to these densities, the radii of discontinuity could be calculated by requiring the models to reproduce the correct total planetary mass and moment of inertia. The latter quantity may be found by the Radau-Darwin formula from the observed ellipticity and rotational velocity of the planets. That such models may lead to erroneous conclusions regarding the matter of which the planets are composed was shown by Ramsey (1951, 1952) and by De Marcus (1958) for the cases of Jupiter and Saturn. It therefore seemed desirable to study the properties of more realistic models of Neptune and Uranus.

Table I. Pressure-density relation for metallic ammonium at 0°K.

Pressure (10^{12} dyne/cm^2)	Density (g/cm^3)
0 00	0.85
0.04	1.28
0.60	1.50
0.80	1.66
1.00	1.80
1.50	2.09
2.00	2.28
2.50	2.44
3.00	2.55
3.50	2.64
4.00	2.70

II. PROPERTIES OF METALLIC AMMONIUM AND EMPIRICAL DATA

In their calculation, Bernal and Massey (1954) first obtained a self-consistent field for the ammonium molecule by averaging the tetrahedral distribution of protons over a sphere of radius r_0. The appropriate value of r_0 was obtained from a calculation of the total energy of the tetrahedral configuration as a function of this radius. The energy of the ammonium metal was then calculated by dividing the metal into spheres of radius a centered on each nitrogen nucleus. By placing the proper requirements on the wave functions of the metallic electron within and on the boundaries of each sphere, the energy per nitrogen atom E was found as a function of a by an iterative procedure. The equation of state of the metal at 0°K was found from the relation

$$p=(3/4\pi)\partial E/\partial(a^3).$$

Their results are given in Table I. These results are applicable to studies of the planets since it is unlikely that their internal temperatures attain values which are high enough to modify appreciably the 0°K isotherm..

In Table II are given the results for the equation of state of solid hydrogen found by De Marcus (1958) and which will be used later in the present work.

The relevant empirical data for Uranus and Neptune are given in Table III. These data are taken from De Marcus (1959), except for the value of $I/M_0R_0^2$ for Uranus. The value of this important quantity is not well known and has been taken to be 0.24 as was done by Wildt (1947). The true value is probably close to this number.

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* Part of this work was performed while the author was a summer visitor with the Theoretical Division, Goddard Space Flight Center, National Aeronautics and Space Administration, Silver Spring, Maryland.

TABLE II. Pressure-density relation for solid hydrogen at 0°K.

Pressure (10^{12} dyne/cm^2)	Density (g/cm^3)
0.00	0.089
0.40	0.399
0.60	0.455
0.80	0.780
1.00	0.82
2.00	1.01
3.00	1.16
4.00	1.29
5.00	1.40
10.00	1.87

III. CALCULATION

Under the assumption of hydrostatic equilibrium in the interior of a cold, rotating body, any pressure p defines an oblate isobaric surface which is approximately a spheroid. If r is the radius of a sphere which has the same volume as the spheroid and if ρ is the density on the surface of this sphere, then, for small rotational velocities ω the equation

$$dp/dr=-\rho(r)[g(r)-\tfrac{2}{3}\omega^2 r] \qquad (1)$$

must hold. Here, $g(r)$ is the local acceleration due to gravity and is given by

$$g(r)=(4\pi G/r^2)\int_0^r \rho(x)x^2dx, \qquad (2)$$

where G is the gravitational constant. It is understood that (1) is to be solved together with the equation of state connecting the pressure and density of the matter of which the planet is hypothesized to be composed. The result of such a solution is the knowledge of the density and pressure as functions of r. The mass distribution may be found from the relation

$$M(r)=(r^2/G)g(r), \qquad (3a)$$

while the moment of inertia may be calculated from

$$I=(8\pi/3)\int_0^{R_0} \rho(x)x^4dx. \qquad (3b)$$

It has been found to be convenient to perform the numerical integration of (1) by beginning at the outer planetary radius and carrying the integration inward to the center. The appropriate starting conditions were

$$\left.\begin{aligned} M&=M_0\\ p&=0\\ \rho&=\rho\ (p=0)\end{aligned}\right\}\text{ at } r=R_0. \qquad (4$$

The mass $M(r)$ should approach zero as r goes to zero If this did not occur, the integration was stopped a $r=0.05\ R_0$ and the remaining mass was ascribed to heavy central "particle" of constant density. Th occurrence of such a central "particle," or core, may b explained qualitatively as due to the settling unde gravity of the heavy elements in the planet since th time of its formation.

Since it was desired to allow for the presence o materials of higher density than ammonium, the density function used in (1) was taken to be

$$\rho(r)=\rho_1(r)[1+\gamma(1-r/R_0)], \qquad (5)$$

where ρ_1 is the density of the ammonium and γ is constant. This form is also in accord with the idea of gravitational settling of the heavier elements. The constant γ has been treated as an adjustable parameter which was chosen so as to give the best agreement between the computed and empirically determined values of I and to imply the smallest core mass.

TABLE IV. Calculated properties of the interior of Uranus assuming presence of ammonium plus heavier materials ($\gamma=0.00$, $I/M_0R_0^2=0.32$).

$s=r/R_0$	ρ (g/cm^3)	M/M_0	p (10^{12} dyne/cm^2)
1.00	0.84	1.000	0.000
0.90	1.07	0.834	0.227
0.80	1.43	0.661	0.531
0.70	1.74	0.489	0.915
0.60	2.01	0.336	1.35
0.50	2.21	0.212	1.80
0.40	2.36	0.123	2.24
0.30	2.48	0.065	2.67
0.20	2.58	0.034	3.13
0.10	2.70	0.023	3.97
0.05	2.72	0.021	5.40

The important characteristics of the best models obtained in this way for Uranus and Neptune are given in Tables IV and V. The results for Neptune are thought to be acceptable but the calculated value of I is too large by 30% for Uranus.

In order to obtain an acceptable model of Uranus, it was necessary to assume the presence of some material of smaller density than ammonium, at least in the outer regions of the planet. Since its equation of state is well known, this material was taken to be solid hydrogen. If ideal mixing is assumed, the density in the presence of both ammonium and hydrogen is given by

$$1/\rho=(1-w)/\rho_1+w/\rho_2, \qquad (6)$$

where ρ_2 is the density and w is the mass fraction of solid hydrogen. The parameter w has been treated as an ad-

TABLE III. Empirical data of Uranus and Neptune.

Planet	Mass $=M_0(10^{27}$ g)	Mean radius $=R_0$(km)	Mean density (g/cm^3)	$I/M_0R_0^2$
Uranus	86.9	23 700	1.56	0.24
Neptune	102.8	21 500	2.47	0.30

stable constant and it has been further assumed that r radii less than r_1, w became zero. Several values of and r_1 permitted acceptable models of Uranus, the ost satisfactory of which is given in Table VI.

Poor results have been obtained by the application of ese assumptions to Neptune.

IV. CONCLUSIONS

It is concluded that acceptable models of Uranus and eptune may be constructed in which metallic ammo- ium is a major constituent. The best case for Neptune rresponds to an ammonium content of 74% by mass, e remaining material being composed of heavier ele- ents. That for Uranus corresponds to an ammonium ntent of 84%, a hydrogen content of 2%, and a avier element content of 14%. It is not considered gnificant that Uranus appears to have retained some ydrogen while Neptune, the more massive body, has ot. It is thought that allowance for the simultaneous esence of materials both more and less dense than nmonium within the interior of Neptune would lead an acceptable model of that planet in which some ydrogen would occur. However, due to the present ck of information concerning the pressure-density lationships of the heavier elements at high pressures, would seem that such a model would involve an un- rtunate number of adjustable parameters.

It may be noted that the masses of the cores of the best models given imply that, in both cases, the central concentration of heavy elements for these planets is approximately twice the mass of the earth.

TABLE V. Calculated properties of the interior of Neptune ssuming presence of ammonium plus heavier materials ($\gamma=1.00$, $/M_0R_0^2=0.30$).

$s=r/R_0$	ρ (g/cm³)	M/M_0	p (10^{12} dyne/cm²)
1.00	0.85	1.000	0.000
0.90	1.34	0.883	0.352
0.80	2.14	0.732	0.970
0.70	2.92	0.558	1.89
0.60	3.58	0.391	3.04
0.50	4.08	0.249	4.31
0.40	4.35	0.144	5.58
0.30	4.62	0.077	6.81
0.20	4.90	0.040	8.18
0.10	5.17	0.026	10.6
0.05	5.30	0.024	14.8

TABLE VI. Calculated properties of the interior of Uranus assuming presence of some solid hydrogen ($w=\frac{1}{3}$, $r_1=0.90R_0$, $I/M_0R_0^2=0.26$).

$s=r/R_0$	ρ (g/cm³)	M/M_0	p (10^{12} dyne/cm²)
1.00	0.22	1.000	0.000
0.90	0.94	0.935	0.097
0.80	1.29	0.783	0.407
0.70	1.70	0.620	0.853
0.60	2.05	0.467	1.43
0.50	2.32	0.339	2.13
0.40	2.54	0.244	2.97
0.30	2.70	0.181	4.07
0.20	2.72	0.148	5.84
0.10	2.76	0.136	10.5
0.05	2.82	0.134	19.5

Methane is known to be present in the atmosphere of the major planets and it would therefore seem desirable to include it in any study of the interiors of the same planets. For this purpose, a calculation of its equation of state is currently under way.

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The Orbit and Ephemeris of Periodic Comet Oterma

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The orbit of periodic comet Oterma has been integrated on an IBM 650 over the forthcoming very close approach to Jupiter. An ephemeris for 1962–66 is provided and so are osculating values of the orbital elements at intervals during this period.

PERIODIC comet Oterma was discovered at the Turku Observatory on 1943 March 27, and prediscovery images were later found on plates taken there in February 1942. Since then, on account of the low eccentricity of its orbit, the comet has been under observation at opposition each year.

Dr. Paul Herget, using the IBM 650 electronic computer at the University of Cincinnati, has applied a differential correction to his earlier orbit (Herget 1947) incorporating 222 observations made mainly at the Turku, Lick, Yerkes, and McDonald observatories between 1942 February 17 and 1959 June 3. Perturbations by the earth, Jupiter, and Saturn were included. The resulting orbital elements, together with osculating elements for the 1950 and 1958 perihelion passages, are given in Table I.

That the orbit is to be drastically changed during a prolonged close approach to Jupiter between 1961 and 1965 has already been noted (Herget 1947; Oterma 1958), and it was also obvious that the programs which are normally used at the Minor Planet Center in Cincinnati to compute perturbations by a variation-of-elements method would therefore be unsuitable in this case. Hence Cowell's method has been programmed and the integration extended over the close approach, using the IBM 650 at Yale University. Perturbations by all the planets, Venus to Saturn, were included. An ephemeris for the period 1962–66 is given in Table II. For 1961 an ephemeris has been provided (Oterma 1960), and it is unlikely that the comet will be observable after early 1966 until (perhaps) shortly before the next perihelion-passage in 1983. The magnitudes in the accompanying ephemeris have been calculated from

$$\text{mag.} = 9.5 + 10\log r + 5\log\rho,$$

but the comet may well be a magnitude fainter tha this.

Table III gives osculating values of the orbita elements and also the distance (ρ_J) of the comet fror Jupiter for various dates. The results agree very closel with those given by Miss Oterma (1958). It is c interest to note that the node regresses almost 100° i one interval of 40 days and that for a period of severa months it actually advances, that there is no additiona passage through perihelion, and that the size of th orbit is greatly increased. Tisserand's criterion (α) ha been evaluated for two dates, one before and one afte the period of the most violent perturbations by Jupiter It was considered in the form

$$\alpha = \frac{1}{a} + \frac{2[a(1-e^2)]^{\frac{1}{2}}}{a'^{\frac{3}{2}}}\cos I,$$

where the inclination I is referred to Jupiter's orbit, anc where a' is the radius of Jupiter's orbit (assumec circular). It was found that $\alpha = 0.583$ in March 196(and $\alpha = 0.576$ in October 1966. The difference is no large in view of the approximations involved, and really rather small when one considers that the respectiv values of the first term alone are 0.251 and 0.139.

It is unfortunate that the comet is in conjunction witl the sun around the time of its closest approach t Jupiter, and that afterwards it will be so extremely faint; but it is to be hoped that observations can stil

Table I. Orbital elements.

Epoch (ET)	1943 Oct. 3.0	1950 July 18.0	1958 June 6.0
T (ET)	1942 Aug. 21.7241	1950 July 16.4856	1958 June 10.5045
ω	354°7890	354°7913	354°8723
Ω 1950.0	155.1711	155.1305	155.1100
i	3.9899	3.9886	3.9921
q	3.389446	3.404882	3.387831
e	0.144443	0.142763	0.144497
a	3.961685	3.971926	3.960045
n°	0.1249925	0.1245095	0.1250702
P (years)	7.89	7.92	7.88

Table II. Ephemeris for 1962–66.

0h ET	α (1950.0)	δ	ρ	r	Mag.
2–63					
r. 17	$22^{h}03^{m}.55$	$-11^{\circ}26'.8$	5.504	4.639	19.9
27	22 12.50	10 41.1			
r. 6	22 21.00	9 56.8	5.343	4.650	19.8
16	22 28.96	9 14.7			
26	22 36.30	8 35.6	5.120	4.661	19.7
y 6	22 42.93	8 00.3			
16	22 48.75	7 29.7	4.852	4.671	19.6
26	22 53.65	7 04.7			
ne 5	22 57.54	6 46.0	4.559	4.682	19.5
15	23 00.31	6 34.6			
25	23 01.87	6 31.0	4.267	4.692	19.4
y 5	23 02.15	6 35.8			
15	23 01.12	6 49.2	4.008	4.703	19.2
25	22 58.80	7 10.8			
g. 4	22 55.29	7 39.8	3.817	4.714	19.1
14	22 50.79	8 14.8			
24	22 45.58	8 53.7	3.725	4.725	19.1
pt. 3	22 40.02	9 33.8			
13	22 34.52	10 12.7	3.751	4.737	19.1
23	22 29.49	10 47.7			
t. 3	22 25.27	11 16.8	3.896	4.749	19.2
13	22 22.16	11 38.7			
23	22 20.32	11 52.5	4.138	4.762	19.4
v. 2	22 19.85	11 57.9			
12	22 20.74	11 55.1	4.444	4.776	19.5
22	22 22.94	11 44.4			
c. 2	22 26.36	11 26.4	4.777	4.792	19.7
12	22 30.86	11 01.8			
22	22 36.30	10 31.1	5.104	4.810	19.9
n. 1	22 42.57	9 55.1			
11	22 49.51	9 14.4	5.398	4.831	20.0
21	22 57.01	8 29.6			
31	23 04.95	− 7 41.4	5.638	4.855	20.1
63–64					
ay 11	0 29.29	+ 1 38.5	5.729	5.025	20.3
21	0 36.66	2 31.0			
31	0 43.54	3 20.3	5.527	5.053	20.2
ne 10	0 49.85	4 05.8			
20	0 55.48	4 46.9	5.276	5.076	20.2
30	1 00.34	5 22.9			
ly 10	1 04.32	5 53.4	4.995	5.094	20.1
20	1 07.33	6 17.9			
30	1 09.27	6 35.9	4.712	5.109	20.0
g. 9	1 10.09	6 47.1			
19	1 09.75	6 51.5	4.456	5.122	19.9
29	1 08.28	6 49.1			
pt. 8	1 05.75	6 40.4	4.262	5.134	19.8
18	1 02.34	6 26.3			
28	0 58.28	6 08.1	4.160	5.145	19.7
t. 8	0 53.88	5 47.4			
18	0 49.48	5 26.3	4.172	5.156	19.7
28	0 45.42	5 06.8			
v. 7	0 42.02	4 50.9	4.300	5.166	19.8
17	0 39.53	4 40.2			
27	0 38.12	4 35.8	4.526	5.177	19.9
c. 7	0 37.88	4 38.3			
17	0 38.84	4 48.0	4.819	5.188	20.1
27	0 40.97	5 04.8			
n. 6	0 44.21	5 28.1	5.144	5.200	20.2
16	0 48.47	5 57.5			
26	0 53.63	+ 6 32.1	5.465	5.211	20.4
1964					
Feb. 5	$0^{h}59^{m}.60$	$+\ 7^{\circ}11'.4$			
15	1 06.26	7 54.4	5.756	5.224	20.5
25	1 13.52	8 40.5			
Mar. 6	1 21.27	+ 9 28.9	5.994	5.237	20.6
1964–65					
June 14	2 48.12	+17 27.1	6.074	5.310	20.7
24	2 56.05	18 04.6			
July 4	3 03.51	18 39.1	5.874	5.326	20.6
14	3 10.38	19 10.3			
24	3 16.56	19 38.0	5.623	5.344	20.5
Aug. 3	3 21.95	20 02.2			
13	3 26.41	20 22.7	5.341	5.361	20.4
23	3 29.83	20 39.3			
Sept. 2	3 32.12	20 51.9	5.053	5.380	20.3
12	3 33.19	21 00.2			
22	3 33.00	21 04.1	4.789	5.398	20.2
Oct. 2	3 31.55	21 03.5			
12	3 28.94	20 58.4	4.584	5.418	20.1
22	3 25.32	20 49.0			
Nov. 1	3 20.95	20 35.8	4.473	5.438	20.1
11	3 16.15	20 19.7			
21	3 11.30	20 01.8	4.478	5.458	20.1
Dec. 1	3 06.76	19 43.7			
11	3 02.87	19 27.0	4.605	5.479	20.2
21	2 59.90	19 13.0			
31	2 58.05	19 02.9	4.836	5.501	20.3
Jan. 10	2 57.42	18 57.5			
20	2 58.05	18 57.2	5.139	5.523	20.5
30	2 59.90	19 02.0			
Feb. 9	3 02.92	19 11.5	5.476	5.546	20.6
19	3 07.01	19 25.2			
Mar. 1	3 12.06	19 42.6	5.813	5.569	20.8
11	3 17.96	20 02.9			
21	3 24.61	20 25.4	6.119	5.592	20.9
31	3 31.89	20 49.4			
Apr. 10	3 39.71	+21 14.3	6.373	5.616	21.0
1965–66					
Aug. 8	5 22.39	+24 52.9	6.314	5.769	21.1
18	5 28.99	25 00.3			
28	5 34.81	25 06.4	6.064	5.796	21.0
Sept. 7	5 39.75	25 11.6			
17	5 43.70	25 16.0	5.783	5.824	21.0
27	5 46.55	25 19.8			
Oct. 7	5 48.21	25 23.2	5.497	5.851	20.9
17	5 48.62	25 26.2			
27	5 47.76	25 28.7	5.241	5.879	20.8
Nov. 6	5 45.68	25 30.4			
16	5 42.48	25 31.1	5.053	5.907	20.7
26	5 38.36	25 30.5			
Dec. 6	5 33.61	25 28.4	4.965	5.936	20.7
16	5 28.55	25 24.6			
26	5 23.56	25 19.5	5.000	5.965	20.8
Jan. 5	5 18.99	25 13.4			
15	5 15.15	25 07.0	5.156	5.994	20.8
25	5 12.30	25 00.8			
Feb. 4	5 10.59	24 55.4	5.413	6.023	21.0
14	5 10.10	24 51.2			
24	5 10.85	24 48.4	5.735	6.053	21.1
Mar. 6	5 12.80	24 47.0			
16	5 15.87	+24 46.8	6.084	6.083	21.3

TABLE III. Osculating values.

Epoch		M	ω	Ω	i	q	e	P	ρ
1960	Mar. 7	78°0	356°4	154°9	4°0	3.394	0.147	$7^{\mathrm{y}}.93$	1.
	Sept. 23	100.9	358.6	154.4	4.0	3.404	0.148	7.98	1.
1961	Apr. 11	121.8	2.9	153.0	4.1	3.416	0.150	8.06	0.
	Oct. 28	139.6	10.0	150.8	4.4	3.412	0.155	8.12	0.
1962	May 16	151.5	19.9	149.4	5.0	3.328	0.174	8.09	0.
	Dec. 2	143.1	40.7	149.4	5.4	3.063	0.232	7.97	0.
1963	Jan. 11	132.5	53.7	146.0	4.6	3.034	0.249	8.13	0.
	Feb. 20	111.8	95.9	121.3	2.6	3.182	0.255	8.83	0.
	Apr. 1	68.4	233.8	25.6	3.9	3.875	0.244	11.6	0.
	May 11	22.9	302.2	12.6	7.0	4.672	0.320	18.0	0.
	June 20	9.8	327.4	12.6	7.0	4.958	0.385	22.9	0.
	July 30	6.4	338.0	13.5	6.2	5.053	0.393	24.0	0.
	Sept. 8	5.5	344.0	14.2	5.4	5.096	0.382	23.7	0.
	Oct. 18	5.6	348.3	14.3	4.7	5.121	0.367	23.0	0.
	Nov. 27	6.1	351.9	13.8	4.1	5.139	0.352	22.4	0.
1964	Jan. 6	7.1	355.2	12.9	3.6	5.154	0.339	21.8	0.
	July 24	14.2	13.5	1.5	2.3	5.215	0.292	20.0	0.
1965	Feb. 9	23.0	33.5	345.6	1.9	5.273	0.268	19.3	0.
	Aug. 28	31.9	46.4	335.9	1.9	5.322	0.256	19.2	0.
1966	Mar. 16	41.0	52.1	332.4	1.9	5.383	0.250	19.2	1.
	Oct. 2	50.4	54.5	331.5	1.9	5.421	0.247	19.3	1.

be made afterwards, perhaps leading to an improved value for the mass of Jupiter.

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